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ESTIMATION OF CHANGE-POINT FOR SWITCHING FRACTIONAL DIFFUSION PROCESSES

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We study the asymptotic distribution of the maximum likelihood estimator for the change point for fractional diffusion processes as the noise intensity tends to zero. It was shown that the rate of convergence here is higher as compared to the rate of convergence of the distribution of the maximum likelihood estimator in classical parametric models dealing with independent identically distributed observations with finite and positive Fisher information.

Keywords: Change point; fractional Brownian motion; fractional diffusion process; estimation.

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1 Introduction

Change-point problems or disorder problems have been of interest to statisticians for their applications and for probabilists for their challenging problems. Recent applications of change-point methods include finance, statistical image processing and edge detection in noisy images which can be considered as a multidimensional change-point and boundary detection problem. Estimation of change-points in economic models such as split or two-phase regression and changes in hazard or failure rates in modelling life times after bone-marrow transplantation of leukemia patients is of practical interest. A study of change-point problems and their applications are discussed in the monograph on change-point problems edited by Carlstein et al. [3]. Csorgo and Horvath [5] discuss limit theorems in change point analysis. Deshayes and Picard [7] study asymptotic distributions of tests and estimators for change point in the classical statistical model of independent observations (cf. Prakasa Rao [25]). The problem of estimation of both the change point and parameters in the drift and diffusion has been considered recently by many authors in continuous as well as discrete time. The disorder problem for diffusion type processes, that is, processes driven by Wiener process,

is investigated in Kutoyants [15], Kutoyants [16] and more recently in Kutoyants [17]. Kutoyants [16] considered the problem of simultaneous estimation of the trend parameter and change point for diffusion type processes. Prakasa Rao [26] gives a comprehensive survey on problems of estimation for diffusion type processes observed over in continuous time or over discrete time. For some recent work on the change point problems for diffusion processes, see Lee et al. [19], Song and Lee [30], De Gregorio and Iacus [6] and Iacus and Yoshida [10],[11].

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao [26]. There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion (fBm) in view of their applications for modeling time series which are long-range dependent. In a recent paper, Kleptsyna and Le Breton [13] studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$(1. 1) \quad X_t = X_0 + \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \rightarrow \infty$. A survey of results on statistical inference for fractional diffusion processes, that is, processes driven by a fractional Brownian motion, is given in Prakasa Rao [27]. For more recent work on parametric estimation for fractional Ornstein-Uhlenbeck process, see Xiao et al. [31], Hu and Nualart [8] and Hu et al. [9].

Our aim in this paper is to consider estimation of the change point τ for a model of fractional diffusion process with small diffusion coefficient. We consider the model

$$(1. 2) \quad dX_t = S_t(\tau, X) dt + \epsilon dW_t^H, X_0 = x_0, 0 \leq t \leq T$$

where $\{W_t^H, 0 \leq t \leq T\}$ is the fractional Brownian motion with known Hurst index $H \in [\frac{1}{2}, 1)$, $S_t(\tau, x) = h_t(x)$ if $t \in [0, \tau]$ and $S_t(\tau, x) = g_t(x)$ if $t \in (\tau, T]$, with $h_t(x)$ and $g_t(x)$ known functions. We estimate the change point τ by the maximum likelihood method ($\hat{\tau}_\epsilon$) and to study its asymptotic properties following the methods in Ibragimov and Khasminskii

[12] and Prakasa Rao [24]. We show that the normalized sequence

$$\epsilon^{-2}(\hat{\tau}_\epsilon - \tau)$$

has a limiting distribution as $\epsilon \rightarrow 0$. We note that the change point problems belong to a class of non-regular statistical problems in the sense that the rate of convergence of the estimator here is higher than the standard rate of convergence of the maximum likelihood estimator of a parameter in the classical case of independent and identically distributed observations with a density function which is twice differentiable and with finite positive Fisher information. This was earlier observed by Chernoff and Rubin [4], Deshayes and Picard [7] in their study of estimation of the change point and by Prakasa Rao [24] in his study of estimation of the location of the cusp of a continuous density. The rate of convergence of the estimator $\hat{\tau}_\epsilon$ observed here is ϵ^2 as $\epsilon \rightarrow 0$.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are (\mathcal{F}_t) -adapted. Further the natural filtration of a process is understood as the P -completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$(2.1) \quad E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s - t|^{2H}], t \geq 0, s \geq 0.$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$(2.2) \quad Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0$$

where $C = \{C(t), t \geq 0\}$ is an (\mathcal{F}_t) -adapted process and $B(t)$ is a non-vanishing non-random function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$(2.3) \quad dY_t = C(t)dt + B(t)dW_t^H, t \geq 0; Y_0 = 0$$

driven by the fractional Brownian motion W^H . The integral

$$(2.4) \quad \int_0^t B(s)dW_s^H$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to a fractional Brownian motion in a natural sense (cf. Norros et al. [21], Alos et al. [1]). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration (\mathcal{Z}_t) of the process Z coincides with the natural filtration (\mathcal{Y}_t) of the process Y (Kleptsyna et al. [14]). Define, for $0 < s < t$,

$$(2.5) \quad k_H = 2H \Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right),$$

$$(2.6) \quad \kappa_H(t, s) = k_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H},$$

$$(2.7) \quad \lambda_H = \frac{2H \Gamma(3-2H)\Gamma\left(H + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2} - H\right)},$$

$$(2.8) \quad w_t^H = \lambda_H^{-1} t^{2-2H},$$

and

$$(2.9) \quad M_t^H = \int_0^t \kappa_H(t, s) dW_s^H, t \geq 0.$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. [21]) and its quadratic variation is given by $\langle M_t^H \rangle = w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H . In fact the stochastic integral

$$(2.10) \quad \int_0^t B(s) dW_s^H$$

can be represented in terms of the stochastic integral with respect to the martingale M^H . For a measurable function f on $[0, T]$, let

$$(2.11) \quad K_H^f(t, s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{1}{2}} dr, 0 \leq s \leq t$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. [29] for sufficient conditions). The following result is due to Kleptsyna et al. [14].

Theorem 2.1: Let M^H be the fundamental martingale associated with the fractional Brownian motion W^H defined by (2.9). Then

$$(2.12) \quad \int_0^t f(s) dW_s^H = \int_0^t K_H^f(t, s) dM_s^H, t \in [0, T]$$

P -a.s. whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. [29]) so that

$$(2.13) \quad Q_H(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) \frac{C(s)}{B(s)} ds, t \in [0, T]$$

is well-defined where w^H and κ_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. The following theorem due to Kleptsyna et al. [14] associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration (\mathcal{Z}_t) of Z coincides with the natural filtration (\mathcal{Y}_t) of Y .

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P -a.s to $L^2([0, T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0, T])$ be defined by

$$(2.14) \quad Z_t = \int_0^t \kappa_H(t, s) B^{-1}(s) dY_s$$

where the function $\kappa_H(t, s)$ is as defined in (2.6). Then the following results hold:

(i) The process Z is an (\mathcal{F}_t) -semimartingale with the decomposition

$$(2.15) \quad Z_t = \int_0^t Q_H(s) dw_s^H + M_t^H$$

where M^H is the fundamental martingale defined by (2.9),

(ii) the process Y admits the representation

$$(2.16) \quad Y_t = \int_0^t K_H^B(t, s) dZ_s$$

where the function K_H^B is as defined in (2.11), and

(iii) the natural filtrations (\mathcal{Z}_t) and (\mathcal{Y}_t) coincide.

Kleptsyna et al. [14] derived the following Girsanov-type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

$$(2.17) \quad \Lambda_H(T) = \exp\left\{-\int_0^T Q_H(t) dM_t^H - \frac{1}{2} \int_0^T Q_H^2(t) dw_t^H\right\}.$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V

defined by

$$(2.18) \quad V_t = \int_0^t B(s) dW_s^H, 0 \leq t \leq T$$

under the probability measure P .

3 Assumptions and main result

Let a process $X = \{X_t, 0 \leq t \leq T\}$ be observed over the time interval $[0, \tau]$ and suppose it follows the stochastic integral equation

$$(3.1) \quad X_t = x_0 + \int_0^t h_u(X) du + \epsilon \int_0^t dW_u^H, t \in [0, \tau]$$

and suppose the process $X^* = \{X^*(t), \tau < t \leq T\}$ is observed over the time interval $(\tau, T]$ satisfying the stochastic integral equation

$$(3.2) \quad X_t^* = X_\tau + \int_\tau^t g_u(X^*) du + \epsilon \int_\tau^t dW_u^H, t \in (\tau, T].$$

where W^H is the fractional Brownian motion with known Hurst parameter $H \in [\frac{1}{2}, 1)$. Let

$$\begin{aligned} S_t(\tau, x) &= h_t(x), \quad 0 \leq t \leq \tau \\ &= g_t(x), \quad \tau < t \leq T. \end{aligned}$$

Suppose that the change point $\tau \in [t_1, t_2] \subset [0, T]$ where t_1 and t_2 are known but arbitrary in the interval $[0, T]$. We assume that the functions $g_t(\cdot)$ and $h_t(\cdot)$ are known but the change point τ is unknown. For convenience, we denote the process X^* by X over the interval $(\tau, T]$. It is required to estimate the change point τ from the realization of X over the interval $[0, T]$. Let $\hat{\tau}_\epsilon$ denote the maximum likelihood estimator (MLE) of τ . We are interested in the asymptotic behaviour of the MLE $\hat{\tau}_\epsilon$ as $\epsilon \rightarrow 0$.

Let $x = \{x(t), 0 \leq t \leq T\}$ with $x(0) = x_0$ be the solution of the ordinary differential equation

$$\begin{aligned} \frac{dx(t)}{dt} &= h_t(x), 0 \leq t \leq \tau \\ &= g_t(x), \tau < t \leq T. \end{aligned}$$

We assume that the trend coefficient $S_t(\tau, X)$ satisfies the following conditions which ensure the existence and the uniqueness of a unique solution of the equation (1.2).

(A.1) There exists a constant $L > 0$ such that

$$|S_t(\tau, X) - S_t(\tau, Y)| \leq L|X_t - Y_t|, 0 \leq t, \tau \leq T.$$

(A.2) There exists a constant $M > 0$ such that $|S_t(\tau, X)| \leq M(1 + |X_t|), t, \tau \in [0, T]$.

The existence and the uniqueness of the solution of the stochastic differential equation (1.2) follow from the results in Nualart and Rascanu [22].

The general method of obtaining the asymptotic properties of the maximum likelihood estimator (MLE) for the change point τ by Taylor's expansion of the log-likelihood is not applicable in this situation due to non-differentiability of the likelihood ratio with respect to the parameter τ . Therefore we follow the technique used by Prakasa Rao [24], Ibragimov and Khasminskii [12], Kutoyants [15] and others. We prove the weak convergence of the appropriately normalized log-likelihood ratio process and appeal to the continuous mapping theorem to study the asymptotic behaviour of the MLE.

Let

$$(3.3) \quad A_t(\tau, x) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) S_s(\tau, x) ds, 0 \leq t \leq T.$$

Consider the transformed processes

$$(3.4) \quad A_t(\tau, X) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s) S_s(\tau, X) ds, 0 \leq t \leq T,$$

$$(3.5) \quad Y_t = \int_0^t \kappa_H(t, s) dX(s), 0 \leq t \leq T$$

and the martingale

$$(3.6) \quad M_t^H = \int_0^t \kappa_H(t, s) dW_s^H, 0 \leq t \leq T.$$

Then the process $Y = \{Y_t, 0 \leq t \leq T\}$ defined by (3.5) satisfies the stochastic differential equation

$$(3.7) \quad dY_t = A_t(\tau, X) dw_t^H + \epsilon dM_t^H, 0 \leq t \leq T$$

where M^H is the fundamental martingale given by (3.6) and Y is a semimartingale (cf. Kleptsyna et al. [14]).

(A.3) Suppose the function $A_t(\tau, x)$ is Lipschitz in x uniformly in τ and t , that is, there exists a constant $C > 0$ independent of τ, t such that

$$|A_t(\tau, x) - A_t(\tau, y)| \leq C|x - y|, 0 \leq t \leq T, 0 \leq t_1 \leq \tau \leq t_2 \leq T.$$

Let

$$(3. 8) \quad \Delta_t = A_t(\tau + \epsilon^2 v, X) - A_t(\tau, X)$$

and

$$(3. 9) \quad \bar{\Delta}_t = A_t(\tau + \epsilon^2 v, x) - A_t(\tau, x)$$

for a given $v \in R$.

(A.4)(i) Let

$$\delta_t = A_t(\tau + \epsilon^2 v_1, X) - A_t(\tau + \epsilon^2 v_2, X)$$

with $\tau + \epsilon^2 v_1 = \theta_1$ and $\tau + \epsilon^2 v_2 = \theta_2$. Suppose there exists a neighbourhood N_τ of τ such that

$$(3. 10) \quad \sup_{\theta_1, \theta_2 \in N_\tau} \sup_{0 \leq t \leq T} E_{\theta_1}(\delta_t^8) < \infty.$$

(A.4)(ii) Suppose that

$$J_\tau = \lim_{\epsilon \rightarrow 0, t \rightarrow \tau, t > \tau} \bar{\Delta}_t = \lim_{t \rightarrow \tau + 0, t > \tau} \frac{d}{dw_t^H} \int_0^t \kappa_H(t, s)(g_s(x) - h_s(x)) I_{[\tau \leq s < \tau + \epsilon^2 v]} ds$$

exists, is independent of $v > 0$, and, for all $v > 0$,

$$\lim_{\epsilon \rightarrow 0} \sup_{[\tau \leq t < \tau + \epsilon^2 v]} |\bar{\Delta}_t - J_t|^2 t^{1-2H} = 0;$$

and

$$\lim_{\epsilon \rightarrow 0} \sup_{[\tau \leq t < \tau + \epsilon^2 v]} |\bar{\Delta}_t^2 - J_t^2| t^{1-2H} = 0.$$

In addition to the conditions (A.1) to (A.4), we assume that the following condition holds:

(A.5) There exist constants $c > 0$ and $C > 0$ possibly depending on H and T such that

$$cv^\beta \leq \epsilon^{-2} \int_\tau^{\tau + \epsilon^2 v} [A_t(\tau + \epsilon^2 v, x) - A_t(\tau, x)]^2 dw_t^H \leq Cv^\beta, \tau \in [t_1, t_2],$$

for some $\beta > 0$.

The filtrations of the transformed process Y and the process X coincide by Theorem 1 of Kleptsyna et al. [14] and hence the problem of estimation of the parameter τ from the process X and the problem of estimation of the parameter τ from the process Y are equivalent. We now consider the problem of estimation of the change point τ based on the observation $\{Y_t, 0 \leq t \leq T\}$ by the method of maximum likelihood. Let τ_ϵ denote the maximum likelihood estimator based on the observation $\{Y_t, 0 \leq t \leq T\}$. Let

$$(3.11) \quad J_\tau^* = J_\tau \sqrt{\tau^{1-2H} \lambda_H^{-1}}.$$

and

$$(3.12) \quad \begin{aligned} L_0(v) &= J_\tau^* W_1(v) - \frac{1}{2} J_\tau^{*2} v \text{ if } v \geq 0 \\ &= J_\tau^* W_2(-v) + \frac{1}{2} J_\tau^{*2} v \text{ if } v < 0 \end{aligned}$$

where $\{W_1(v), v \geq 0\}$ and $\{W_2(v), v \geq 0\}$ are independent standard Wiener processes.

We now state the main result of this paper.

Theorem 3.1: Suppose the conditions (A.1) – (A.5) hold. Let τ denote the true change point and $\hat{\tau}_\epsilon$ denote the maximum likelihood estimator of τ based on the observation of the process X satisfying the stochastic differential system defined by (3.1) and (3.2). Then, as $\epsilon \rightarrow 0$, the normalized random variable

$$\epsilon^{-2}(\hat{\tau}_\epsilon - \tau)$$

converges in law to a random variable ψ whose distribution is the distribution of location of the maximum of the process $\{L_0(v), -\infty < v < \infty\}$ as defined above.

4 Weak convergence of the log-likelihood ratio process

At first, we state a lemma which gives upper bounds on the difference $X_t - x_t$ and $E_\tau(X_t - x_t)^2$ where the process $\{X_t, 0 \leq t \leq T\}$ satisfies the stochastic differential equation system defined by the equations (3.1) and (3.2) and the function x_t is the solution of the corresponding ordinary differential equation discussed above. Throughout the following discussion, we shall denote an arbitrary positive constant by C which might be different from one inequality to the other.

Lemma 4.0: Let the trend function $S_\tau(t, x)$ satisfy the conditions (A.1) and (A.2). Then, with probability one,

$$(i) |X_t - x_t| \leq e^{Lt} \epsilon |W_t^H|$$

and

$$(ii) \sup_{0 \leq t \leq T} E_\tau (X_t - x_t)^2 \leq e^{2LT} \epsilon^2 T^{2H}.$$

For the proof of Lemma 4.0, see Prakasa Rao [27], p. 131.

In view of Lemma 4.0 and the condition (A.3), it follows that there exists a constant C such that

$$(4. 1) \quad \sup_{t_1 \leq \tau' \leq t_2} \sup_{0 \leq t \leq T} E_\tau [A_t(\tau', X) - A_t(\tau', x)]^2 \leq C\epsilon^2.$$

for $\tau \in [t_1, t_2]$.

In particular, it follows that

$$(4. 2) \quad \sup_{0 \leq t \leq T} E_\tau |\Delta_t - \bar{\Delta}_t|^2 \leq C\epsilon^2, \tau \in [t_1, t_2].$$

Let P_τ be the probability measure generated by the process Y on the space $C[0, T]$ associated with the uniform topology when τ is the true change point. Consider the log-likelihood ratio process

$$(4. 3) \quad \begin{aligned} L_\epsilon(v) &= \log \frac{dP_{\tau+\epsilon^2 v}}{dP_\tau} \\ &= \frac{1}{\epsilon} \int_0^T [A_t(\tau + \epsilon^2 v, X) - A_t(\tau, X)] dM_t^H \\ &\quad - \frac{1}{2\epsilon^2} \int_0^T [A_t(\tau + \epsilon^2 v, X) - A_t(\tau, X)]^2 dw_t^H \end{aligned}$$

for fixed v such that $0 \leq \tau, \tau + \epsilon^2 v \leq T$.

Theorem 4.1: Let $-\infty < a < b < \infty$. Under the conditions (A.1) to (A.4), the family of probability measures generated by the log-likelihood ratio processes $\{L_\epsilon(v), v \in [a, b]\}$ on $C[a, b]$, associated with the uniform norm topology, converge weakly to the probability measure generated by the process $\{L_0(v), v \in [a, b]\}$ on $C[a, b]$ as $\epsilon \rightarrow 0$.

From the general theory of weak convergence of probability measures on $C[a, b]$ associated with uniform norm topology (cf. Billingsley [2], Parthasarathy [23], Prakasa Rao [25]), in

order to prove Theorem 4.1, it is sufficient to prove that the finite dimensional distributions of the process $\{L_\epsilon(v), a \leq v \leq b\}$ converge to the corresponding finite dimensional distributions of the process $\{L_0(v), a \leq v \leq b\}$ and the family of probability measures generated by the processes $\{L_\epsilon(v), a \leq v \leq b\}$ for different ϵ is tight.

5 Proof of Theorem 4.1

Before we give a proof of Theorem 4.1, we prove some related lemmas.

Lemma 5.1: Under the conditions (A.1) – (A.3), the the finite distributions of the process $\{L_\epsilon(v), a \leq v \leq b\}$ converge to the corresponding finite dimensional distributions of the process $\{L_0(v), a \leq v \leq b\}$ as $\epsilon \rightarrow 0$.

Proof: We will first investigate the convergence of the one-dimensional marginal distributions of the process $L_\epsilon(v)$ as $\epsilon \rightarrow 0$. The convergence of other classes of finite dimensional distributions follows from the Cramer-Wold device.

Suppose $v > 0$. It can be seen that $\Delta_t = 0$ and $\bar{\Delta}_t = 0$ for $t \in [0, \tau)$ and $t \in [\tau + \epsilon^2 v, T]$. Note that

$$\begin{aligned} L_\epsilon(v) &= \frac{1}{\epsilon} \int_\tau^{\tau+\epsilon^2 v} \Delta_t dM_t^H - \frac{1}{2\epsilon^2} \int_\tau^{\tau+\epsilon^2 v} \Delta_t^2 dw_t^H \\ &= \frac{1}{\epsilon} \int_\tau^{\tau+\epsilon^2 v} (\Delta_t - \bar{\Delta}_t) dM_t^H + \frac{1}{\epsilon} \int_\tau^{\tau+\epsilon^2 v} \bar{\Delta}_t dM_t^H \\ &\quad - \frac{1}{2\epsilon^2} \int_\tau^{\tau+\epsilon^2 v} \Delta_t^2 dw_t^H \\ &= I_1 + I_2 + I_3 \quad (\text{say}). \end{aligned}$$

Now, for any $\epsilon_1 > 0$,

$$\begin{aligned} P(|I_1| \geq \epsilon_1) &\leq \frac{1}{\epsilon_1^2 \epsilon^2} \int_\tau^{\tau+\epsilon^2 v} E_\tau |\Delta_t - \bar{\Delta}_t|^2 dw_t^H \\ &\leq \frac{C}{\epsilon_1^2 \epsilon^2} \sup_{0 \leq t \leq T} E_\tau |\Delta_t - \bar{\Delta}_t|^2 \int_\tau^{\tau+\epsilon^2 v} t^{1-2H} dt \\ &\leq \frac{C}{\epsilon_1^2 \epsilon^2} [(\tau + \epsilon^2 v)^{2-2H} - \tau^{2-2H}] \sup_{0 \leq t \leq T} E_\tau |\Delta_t - \bar{\Delta}_t|^2 \\ &\leq \frac{C}{\epsilon_1^2 \epsilon^2} [(\tau + \epsilon^2 v)^{2-2H} - \tau^{2-2H}] \epsilon^2 \quad (\text{by (A.3)}) \end{aligned}$$

for some positive constant C depending on H and T . Hence $I_1 \xrightarrow{p} 0$ as $\epsilon \rightarrow 0$. Observe that

$$\begin{aligned}
I_2 &= \frac{1}{\epsilon} \int_{\tau}^{\tau+\epsilon^2 v} \bar{\Delta}_t dM_t^H \\
&= \frac{1}{\epsilon} \int_{\tau}^{\tau+\epsilon^2 v} (\bar{\Delta}_t - J_{\tau}) dM_t^H + \frac{1}{\epsilon} \int_{\tau}^{\tau+\epsilon^2 v} J_{\tau} dM_t^H \\
&= \frac{1}{\epsilon} \int_{\tau}^{\tau+\epsilon^2 v} (\bar{\Delta}_t - J_{\tau}) dM_t^H + \frac{1}{\epsilon} J_{\tau} (M_{\tau+\epsilon^2 v}^H - M_{\tau}^H).
\end{aligned}$$

Note that

$$\begin{aligned}
E_{\tau} \left| \frac{1}{\epsilon} \int_{\tau}^{\tau+\epsilon^2 v} (\bar{\Delta}_t - J_{\tau}) dM_t^H \right|^2 &= \frac{1}{\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} |\bar{\Delta}_t - J_{\tau}|^2 dw_t^H \\
&= C \frac{1}{\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} |\bar{\Delta}_t - J_{\tau}|^2 t^{1-2H} dt \\
&= C |\bar{\Delta}_{\eta} - J_{\tau}|^2 \eta^{1-2H}
\end{aligned}$$

for some constant $C > 0$ and for some η such that $\tau < \eta < \tau + \epsilon^2 v$ by the mean value theorem. The last term tends to zero as $\epsilon \rightarrow 0$ by the condition (A.3). Thus the random variable I_2 has the Gaussian distribution with mean zero and variance

$$\begin{aligned}
&\frac{1}{\epsilon^2} E_{\tau} \left(\int_{\tau}^{\tau+\epsilon^2 v} \bar{\Delta}_t dM_t^H \right)^2 \\
&= \frac{1}{\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} \bar{\Delta}_t^2 dw_t^H \\
&= \frac{1}{\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} (\bar{\Delta}_t^2 - J_{\tau}^2) dw_t^H + \frac{1}{\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} J_{\tau}^2 dw_t^H \\
&= v \lambda_H^{-1} J_{\tau}^2 \tau^{1-2H} + o(1).
\end{aligned}$$

The equality stated above follows again by an application of the mean value theorem under the condition (A.3). Furthermore

$$\begin{aligned}
I_3 &= -\frac{1}{2\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} \Delta_t^2 dw_t^H \\
&= -\frac{1}{2\epsilon^2} \left[\int_{\tau}^{\tau+\epsilon^2 v} (\Delta_t - \bar{\Delta}_t + \bar{\Delta}_t)^2 dw_t^H \right] \\
&= -\frac{1}{2\epsilon^2} \left[\int_{\tau}^{\tau+\epsilon^2 v} \left\{ (\Delta_t - \bar{\Delta}_t)^2 + \bar{\Delta}_t^2 + 2(\Delta_t - \bar{\Delta}_t)\bar{\Delta}_t \right\} dw_t^H \right]
\end{aligned}$$

$$\begin{aligned}
&= o_p(1) - \frac{1}{2\epsilon^2} \int_{\tau}^{\tau+\epsilon^2 v} \overline{\Delta}_t^2 dw_t^H \\
&= o_p(1) - \frac{1}{2} v \lambda_H^{-1} J_{\tau}^{*2} \tau^{1-2H}
\end{aligned}$$

as $\epsilon \rightarrow 0$, again, by the mean value theorem under the condition (A.3). As a consequence of the above computations, we get that the random variable $L_{\epsilon}(v)$ is asymptotically Gaussian with the mean $-\frac{1}{2}J_{\tau}^{*2}v$ and the variance $J_{\tau}^{*2}v$. Similar analysis of the above results can be done for the case $v < 0$. This completes the proof of Lemma 5.1.

Remarks: We have proved the convergence of the univariate distributions of the process $\{L_{\epsilon}(v), a \leq v \leq b\}$ as $\epsilon \rightarrow 0$, after proper scaling of the process. Convergence of all the other finite dimensional distributions of the process $\{L_{\epsilon}(v), a \leq v \leq b\}$ as $\epsilon \rightarrow 0$, after proper scaling, follows by an application of the Cramer-Wold device. It can be checked that the covariance matrix of the limiting distribution of $(L_{\epsilon}(v_1), L_{\epsilon}(-v_2))$ for $v_1 > 0, v_2 > 0$, as $\epsilon \rightarrow 0$, after proper scaling will be a diagonal matrix. Since it is the covariance matrix of a bivariate normal distribution, it will in turn imply the independence of the standard Wiener processes W_1 and W_2 in the definition of the limiting process L_0 given by (3.1).

We now state two lemmas which will be used in the following computations.

Lemma 5.2: Let $\{D_t, 0 \leq t \leq T\}$ be a random process such that $\sup_{0 \leq t \leq T} E(D_t^4) \leq \gamma < \infty$. Then, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$E\left(\left[\int_{\theta_2}^{\theta_1} D_t dt\right]^4\right) \leq |\theta_1 - \theta_2|^3 \int_{\theta_2}^{\theta_1} E[D_t^4] dt \leq \gamma |\theta_1 - \theta_2|^4.$$

The inequality given above is an easy consequence of the Holder inequality.

The next lemma gives an inequality for the 4-th moment of a stochastic integral with respect to a martingale which is also of independent interest.

Lemma 5.3: Let the process $\{f_t, 0 \leq t \leq T\}$ be a random process adapted to a square integrable martingale $\{M_t, \mathcal{F}_t, t \geq 0\}$ with the quadratic variation $\langle M \rangle_t$ such that

$$\int_0^T E(f_s^4) d\langle M \rangle_s < \infty.$$

Then

$$E\left(\left(\int_0^T f_t dM_t\right)^4\right) \leq 36 \langle M \rangle_T \int_0^T E(f_t^4) d\langle M \rangle_t.$$

and, in general, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$E\left[\left(\int_{\theta_2}^{\theta_1} f_t dM_t\right)^4\right] \leq 36(\langle M \rangle_{\theta_1} - \langle M \rangle_{\theta_2}) \int_{\theta_2}^{\theta_1} E[f_t^4] d\langle M \rangle_t .$$

Proof : We prove the first part of the lemma. The second part is an easy consequence of the first part. Let

$$\psi(t) = \int_0^t f_s dM_s .$$

Applying the Ito-type lemma for stochastic integrals with respect to martingales for the function $g(\psi(t)) = (\psi(t))^4$, it follows that

$$\left(\int_0^t f_s dM_s\right)^4 = 4 \int_0^t \left(\int_0^s f_u dM_u\right)^4 f_s dM_s + 6 \int_0^t \left(\int_0^s f_u dM_u\right)^2 f_s^2 d\langle M \rangle_s .$$

Taking expectations on both sides of the above equation, we get that

$$(5.1) \quad E\left(\int_0^T f_t dM_t\right)^4 = 6 \int_0^T E\left(\left(\int_0^s f_u dM_u\right)^2 f_s^2\right) d\langle M \rangle_s .$$

Hence

$$(5.2) \quad E\left(\int_0^T f_t dM_t\right)^4 \leq 6\left[\int_0^T E\left(\int_0^s f_u dM_u\right)^4 d\langle M \rangle_s\right]^{1/2} \left(\int_0^T E[f_s^4] d\langle M \rangle_s\right)^{1/2}$$

by an application of the Cauchy-Schwarz inequality. It is clear from the equation (5.1) that the function

$$E\left(\int_0^t f_s dM_s\right)^4$$

is a non-decreasing function of t and hence

$$\int_0^T E\left(\int_0^s f_u dM_u\right)^4 d\langle M \rangle_s \leq \int_0^T E\left(\int_0^T f_u dM_u\right)^4 d\langle M \rangle_s .$$

Using this bound to estimate the first term on the right side of the inequality (5.2), it follows that

$$E\left(\int_0^T f_s dM_s\right)^4 \leq 6(\langle M \rangle_T - \langle M \rangle_0) E\left(\int_0^T f_s dM_s\right)^4)^{1/2} \left(\int_0^T E(f_s^4) d\langle M \rangle_s\right)^{1/2} .$$

From this equation, it now follows that

$$E\left(\int_0^T f_s dM_s\right)^4 \leq 36(\langle M \rangle_T - \langle M \rangle_0) \left(\int_0^T E(f_s^4) d\langle M \rangle_s\right) .$$

Lemma 5.4: Let $\Gamma_\epsilon(v) = \exp L_\epsilon(v)$. Suppose the conditions (A.1) – (A.4) hold. Then, for any compact set $K \subset [0, T]$ there exist constants $C > 0$ depending on H and T such that

$$\sup_{\tau \in K} E_\tau \left| \Gamma_\epsilon^{\frac{1}{4}}(v_2) - \Gamma_\epsilon^{\frac{1}{4}}(v_1) \right|^4 \leq C[(v_1 - v_2)^4 + (v_1 - v_2)^2], a \leq v_1, v_2 \leq b .$$

Proof : Without loss of generality, let $v_1 > v_2$,

$$\delta_t = A_t(\tau + \epsilon^2 v_1, X) - A_t(\tau + \epsilon^2 v_2, X)$$

and

$$\bar{\delta}_t = A_t(\tau + \epsilon^2 v_1, x) - A_t(\tau + \epsilon^2 v_2, x).$$

Let $\tau + \epsilon^2 v_1 = \theta_1$ and $\tau + \epsilon^2 v_2 = \theta_2$. Let

$$R_t = \exp\left[\frac{1}{4\epsilon} \int_0^t \delta_s dM_s^H - \frac{1}{8\epsilon^2} \int_0^t \delta_s^2 dw_s^H\right], R_0 = 1.$$

Note that the process R_t is the process $\left(\frac{dP_{\theta_1}}{dP_{\theta_2}}(X)\right)^{\frac{1}{4}}$ and, by the Ito formula, we have

$$dR_t = -\frac{3}{32\epsilon^2} \delta_t^2 R_t dw_t^H + \frac{1}{4\epsilon} \delta_t R_t dM_t^H.$$

Hence

$$R_T = 1 - \frac{3}{32\epsilon^2} \int_0^T \delta_t^2 R_t dw_t^H + \frac{1}{4\epsilon} \int_0^T \delta_t R_t dM_t^H.$$

Note that

$$\begin{aligned} & E_\tau \left| \Gamma_\epsilon^{\frac{1}{4}}(v_2) - \Gamma_\epsilon^{\frac{1}{4}}(v_1) \right|^4 \\ &= E_\tau \left(\frac{dP_{\theta_2}}{dP_\theta} |1 - R_T|^4 \right) = E_{\theta_2} (|1 - R_T|^4) \\ &\leq C \frac{1}{\epsilon^8} E_{\theta_2} \left| \int_0^T \delta_t^2 R_t dw_t^H \right|^4 + C \frac{1}{\epsilon^4} E_{\theta_2} \left| \int_0^T \delta_t R_t dM_t^H \right|^4 \end{aligned}$$

where C is an absolute constant. In order to get the bounds for the expectations of the integrals in the above inequality, we now use the Lemmas 5.2 and 5.3.

As a consequence of the Lemma 5.2, it follows that there exists a positive constant C depending on H and T such that

$$\begin{aligned} & E_{\theta_2} \left| \frac{1}{\epsilon^2} \int_0^T \delta_t^2 R_t dw_t^H \right|^4 \\ &= E_{\theta_2} \left[\frac{1}{\epsilon^2} \int_0^{\theta_2} \delta_t^2 R_t dw_t^H + \frac{1}{\epsilon^2} \int_{\theta_1}^T \delta_t^2 R_t dw_t^H + \frac{1}{\epsilon^2} \int_{\theta_2}^{\theta_1} \delta_t^2 R_t dw_t^H \right]^4 \end{aligned}$$

$$\begin{aligned}
&= E_{\theta_2} \left| \frac{1}{\epsilon^2} \int_{\theta_2}^{\theta_1} \delta_t^2 R_t dw_t^H \right|^4 \\
&= E_{\theta_2} \left| \frac{1}{\epsilon^2} \int_{\theta_2}^{\theta_1} \delta_t^2 R_t \lambda_H^{-1} (2-2H) t^{1-2H} dt \right|^4 \\
&\leq C \frac{1}{\epsilon^8} (\theta_1 - \theta_2)^3 \int_{\theta_2}^{\theta_1} E_{\theta_2} [\delta_t^2 R_t]^4 t^{4-8H} dt.
\end{aligned}$$

Now

$$\sup_{0 \leq t \leq T} E_{\theta_2} ([\delta_t^2 R_t]^4) = \sup_{0 \leq t \leq T} E_{\theta_1} (\delta_t^8) < \infty$$

by the condition (A.4) since

$$R_t = \left(\frac{dP_{\theta_1}}{dP_{\theta_2}}(X) \right)^{1/4}.$$

Hence

$$E_{\theta_2} \left| \frac{1}{\epsilon^2} \int_0^T \delta_t^2 R_t dw_t^H \right|^4 \leq \frac{C}{\epsilon^8} \gamma |\theta_1 - \theta_2|^3 |\theta_1^{5-8H} - \theta_2^{5-8H}|$$

by the condition (A.4).

Note that, for any $v_1, v_2 \in [a, b]$,

$$\begin{aligned}
&E_{\theta} \left| \Gamma_{\epsilon}^{\frac{1}{4}}(v_2) - \Gamma_{\epsilon}^{\frac{1}{4}}(v_1) \right|^4 \\
&\leq C \frac{1}{\epsilon^8} E_{\theta_2} \left| \int_0^T \delta_t^2 R_t dw_t^H \right|^4 + C \frac{1}{\epsilon^4} E_{\theta_2} \left| \int_0^T \delta_t R_t dM_t^H \right|^4 \\
&\quad \text{(by Lemma 3.3)} \\
&\leq C \frac{1}{\epsilon^8} (\theta_1 - \theta_2)^3 [\theta_1^{5-8H} - \theta_2^{5-8H}] \\
&\quad + C \frac{1}{\epsilon^4} (\theta_1 - \theta_2) [\theta_1^{2-2H} - \theta_2^{2-2H}]^2 \\
&\leq C \frac{1}{\epsilon^8} \epsilon^6 (v_1 - v_2)^3 \tau^{4-8H} \epsilon^2 (v_1 - v_2) \\
&\quad + C \frac{1}{\epsilon^4} \tau^{2-4H} [\epsilon^2 (v_1 - v_2)]^2 \\
&\leq C [(v_1 - v_2)^4 + (v_1 - v_2)^2]
\end{aligned}$$

where C is a positive constant depending on H and T . This completes the proof of Lemma 5.4.

As a consequence of the Lemma 5.4, it follows that the family of probability measures generated by the processes $\{\Gamma_{\epsilon}^{\frac{1}{4}}(v), a \leq v \leq b\}$ on $C[a, b]$ with uniform topology is tight from

the results in Billingsley [2] and hence the family of probability measures generated by the processes $\{L_\epsilon(v), a \leq v \leq b\}$ on $C[a, b]$ is tight.

We will now prove a maximal inequality for the fractional Brownian motion which we will be used later in this discussion. This inequality is a consequence of the Slepian's lemma given below (cf. Matsui and Shieh [20]). For a proof of this lemma, see Leadbetter et al. [18].

Lemma 5.5 : Let $\{\psi_1(t), t \geq 0\}$ and $\{\psi_2(t), t \geq 0\}$ be Gaussian processes with continuous sample paths with $E[\psi_1(t)] = E[\psi_2(t)] = 0$ and $Var(\psi_1(t)) = Var(\psi_2(t)) = 1$ for all t . Let $r_i(t, s)$ denote the covariance function of the process $\{\psi_i(t), t \geq 0\}$ for $i = 1, 2$. Suppose that, for some $d > 0$, $r_1(t, s) \geq r_2(t, s)$ for $0 \leq t, s \leq d$. Let $M_i(t) = \max_{0 \leq s \leq t} \psi_i(s)$, $i = 1, 2$. Then

$$P[M_1(t) \leq u] \geq P[M_2(t) \leq u], u \in R$$

for $0 \leq t \leq d$.

Let $\{\hat{W}_t^H, t \geq 0\}$ be a Gaussian Markov process with independent increments such that $E[\hat{W}_t^H] = 0$ and $Cov(\hat{W}_s^H, \hat{W}_t^H) = s^{2H}$ whenever $0 \leq s \leq t$. From the Slepian's lemma stated above, it follows that

$$P(\max_{0 \leq t \leq T} W_t^H \geq u) \leq P(\max_{0 \leq t \leq T} \hat{W}_t^H \geq u)$$

for all $u \in R$. Hence, by applying the reflection principle for Gaussian Markov process (cf. Revuz and Yor (1999)), it follows that

$$\begin{aligned} P(\max_{0 \leq t \leq T} W_t^H \geq u) &\leq P(\max_{0 \leq t \leq T} \hat{W}_t^H \geq u) \\ &\leq 2 P(\hat{W}_T^H \geq u) \end{aligned}$$

for all $u \in R$.

Note that, from the symmetry of the fractional Brownian motion, it follows that

$$\begin{aligned} P[\max_{0 \leq t \leq T} W_t^H \geq u] &= P[-\max_{0 \leq t \leq T} W_t^H \leq -u] \\ &= P[\min_{0 \leq t \leq T} -W_t^H \leq -u] \\ &= P[\min_{0 \leq t \leq T} W_t^H \leq -u] \end{aligned}$$

and hence

$$\begin{aligned}
P[\max_{0 \leq t \leq T} |W_t^H| \geq u] &\leq P[\max_{0 \leq t \leq T} W_t^H \geq u] + P[\min_{0 \leq t \leq T} W_t^H \leq -u] \\
&= 2 P[\max_{0 \leq t \leq T} W_t^H \geq u] \\
&\leq 2 P[\max_{0 \leq t \leq T} \hat{W}_t^H \geq u] \\
&\leq 4P[\hat{W}_T^H \geq u]
\end{aligned}$$

from the earlier remarks.

Lemma 5.6: For any $\lambda > 0$,

$$E[\exp\{\lambda \max_{0 \leq t \leq T} |W_t^H|\}] \leq 1 + \lambda \sqrt{2\pi T^{2H}} \exp\left\{\frac{\lambda^2 T^{2H}}{2}\right\}.$$

Proof: Let $F(x)$ denote the distribution function of the random variable $W_T^{*H} = \sup_{0 \leq t \leq T} |W_t^H|$. Let $\{\hat{W}_t^H, t \geq 0\}$ be the Gaussian Markov process with independent increments constructed above. Note that $E(\hat{W}_T^H) = 0$ and $Var(\hat{W}_T^H) = T^{2H}$. Let ψ denote a Gaussian random variable with mean zero and variance one. Then, for any $\lambda > 0$,

$$\begin{aligned}
E[\exp\{\lambda \sup_{0 \leq t \leq T} |W_t^H|\}] &= \int_0^\infty e^{\lambda x} dF(x) \\
&= - \int_0^\infty e^{\lambda x} d(1 - F(x)) \\
&= 1 + \lambda \int_0^\infty e^{\lambda x} (1 - F(x)) dx \\
&= 1 + \lambda \int_0^\infty e^{\lambda x} P[W_T^{*H} > x] dx \\
&\leq 1 + 4\lambda \int_0^\infty e^{\lambda x} P[\hat{W}_T^H > x] dx \\
&= 1 + 4\lambda \int_0^\infty e^{\lambda x} P[\psi T^H > x] dx \\
&= 1 + 4\lambda \int_0^\infty e^{\lambda x} P[\psi > x T^{-H}] dx \\
&\leq 1 + 4\lambda \int_0^\infty e^{\lambda x} \frac{1}{2} \exp\left\{-\frac{x^2 T^{-2H}}{2}\right\} dx \\
&\leq 1 + \lambda \sqrt{2\pi T^{2H}} \exp\left\{\frac{T^{2H} \lambda^2}{2}\right\}
\end{aligned}$$

We now apply Lemma 5.6 to get the following result.

Lemma 5.7: Suppose the conditions (A.1) to (A.5) hold. Let $\Gamma_\epsilon(v) = \exp\{L_\epsilon(v)\}$, $v \in R$. Then, for any compact set $K \subset [0, T]$, and for any $0 < p < 1$, there exists a positive constant C such that

$$(5.3) \quad \sup_{\tau \in K} E_\tau[(\Gamma_\epsilon(v))^p] \leq e^{-C|v|^\beta}$$

for all v .

Proof: Now, for any $0 < p < 1$, we will now estimate $E_\tau(\Gamma_\epsilon(v))^p$. For convenience, let $v > 0$ and let

$$F_1 \equiv \int_\tau^{\tau+\epsilon^2 v} \Delta_t dM_t^H$$

and

$$F_2 \equiv \int_\tau^{\tau+\epsilon^2 v} \Delta_t^2 dw_t^H.$$

Let q be such that $p^2 < q < p$. Then

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(v))^p] &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{p}{2\epsilon^2}F_2\}] \\ &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2 - \frac{(p-q)}{2\epsilon^2}F_2\}]. \end{aligned}$$

Let

$$G_1 = \exp\{-\frac{(p-q)}{2\epsilon^2}F_2\}$$

and

$$G_2 = \exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2\}.$$

Then

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(v))^p] &= E_\tau[G_1 G_2] \\ &\leq (E[G_1^{p_1}])^{1/p_1} (E[G_2^{p_2}])^{1/p_2} \end{aligned}$$

by the Holder inequality for any p_1 and p_2 such that $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Choose $p_2 = \frac{q}{p^2} > 1$. Then $p_1 = \frac{q}{q-p^2}$. Observe that

$$\begin{aligned} E_\tau[G_2^{p_2}] &= E_\tau[\exp\{p_2(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \\ &= E_\tau[\exp\{\frac{q}{p^2}(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \end{aligned}$$

$$= E_\tau[\exp\{\frac{1}{\epsilon} \frac{q}{p} F_1 - \frac{1}{2\epsilon^2} \frac{q^2}{p^2} F_2\}].$$

The random variable under the expectation sign in the last line is the Radon-Nikodym derivative of two probability measures which are absolutely continuous with respect to each other by the Girsanov's theorem for martingales. Hence the expectation is equal to one. Hence

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(v))^p] &\leq (E[\exp\{-\frac{p_1(p-q)}{2\epsilon^2} F_2\}])^{1/p_1} \\ &= (E[\exp\{-\gamma\epsilon^{-2} F_2\}])^{1/p_1}. \end{aligned}$$

where $\gamma = \frac{q(p-q)}{2(q-p^2)} > 0$. Let us now estimate $E[e^{-\gamma\epsilon^{-2} F_2}]$. Applying the inequality

$$a^2 \geq b^2 - 2|b(a-b)|,$$

it follows that

$$\begin{aligned} E[e^{-\gamma\epsilon^{-2} F_2}] &\leq \exp\{-\gamma\epsilon^{-2} \int_\tau^{\tau+\epsilon^2 v} \bar{\Delta}_t^2 dw_t^H\} \times \\ &\quad \times E_\tau[\exp\{2\gamma\epsilon^{-2} (\int_\tau^{\tau+\epsilon^2 v} (A_t(\tau+\epsilon^2 v, X) - A_t(\tau+\epsilon^2 v, x)) + \\ &\quad + |A_t(\tau, X) - A_t(\tau, x)|) |A_t(\tau+\epsilon^2 v, x) - A_t(\tau, x)| dw_t^H\}]. \end{aligned}$$

We now get an upper bound on the term under the expectation sign on the right side of the above inequality. Observe that there exists a constant $L > 0$, such that,

$$\begin{aligned} \int_\tau^{\tau+\epsilon^2 v} [A_t(\tau, X) - A_t(\tau, x)]^2 dw_t^H &\leq \int_0^T [A_t(\tau, X) - A_t(\tau, x)]^2 dw_t^H \\ &\leq L^2 [\int_0^T |X_t - x_t|^2 dw_t^H] \\ &\leq L^2 \epsilon^2 \sup_{0 \leq t \leq T} |X_t - x_t|^2 [\int_0^T dw_t^H] \\ &\leq L^2 \epsilon^2 e^{2LT} [\int_0^T dw_t^H] \sup_{0 \leq t \leq T} |W_t^H|^2 \text{ (by Lemma 4.0)} \\ &\leq C \epsilon^2 L^2 e^{2LT} T^{2-2H} \sup_{0 \leq t \leq T} |W_t^H|^2. \end{aligned}$$

for some constant $C > 0$ possibly depending on T and H . Therefore, under the condition (A.5), for any $\tau' \in [t_1, t_2]$, an application of Cauchy-Schwarz inequality implies that

$$\sup_{t_1 \leq \tau, \tau' \leq t_2} [\int_\tau^{\tau+\epsilon^2 v} |A_t(\tau+\epsilon^2 v, x) - A_t(\tau, x)| |A_t(\tau', X) - A_t(\tau', x)| dw_t^H]^2$$

$$\leq CL^2 \epsilon^4 v^\beta e^{2LT} T^{2-2H} \sup_{0 \leq t \leq T} |W_t^H|^2.$$

Hence

$$\begin{aligned} \sup_{t_1 \leq \tau, \tau' \leq t_2} \left[\int_{\tau}^{\tau + \epsilon^2 v} |A_t(\tau + \epsilon^2 v, x) - A_t(\tau, x)| |A_t(\tau', X) - A_t(\tau', x)| dw_t^H \right] \\ \leq C \epsilon^2 v^{\frac{\beta}{2}} \sup_{0 \leq t \leq T} |W_t^H|. \end{aligned}$$

Therefore

$$\begin{aligned} \sup_{t_1 \leq \tau \leq t_2} E_{\tau} [\exp\{2\gamma \epsilon^{-2} (\int_{\tau}^{\tau + \epsilon^2 v} (|A_t(\tau + \epsilon^2 v, X) - A_t(\tau + \epsilon^2 v, x)| + \\ + |A_t(\tau, X) - A_t(\tau, x)|) |A_t(\tau + \epsilon^2 v, x) - A_t(\tau, x)| dw_t^H)\}] \\ \leq E_{\tau} [\exp\{C \gamma v^{\frac{\beta}{2}} \sup_{0 \leq t \leq T} |W_t^H|\}] \\ = 1 + \gamma C v^{\frac{\beta}{2}} \sqrt{2\pi T^{2H}} \exp\left\{\frac{\gamma^2 c_6^2 T^{2H} v^{\beta}}{2}\right\} \end{aligned}$$

by Lemma 5.6. Applying arguments similar to those in Lemma 2.4 in Kutoyants (1994), we get that

$$\sup_{\tau \in K} E_{\tau} [\Gamma_{\epsilon}^p(v)] \leq e^{-Cv^{\beta}}$$

for some positive constant $C > 0$ depending on T and H .

An application, of the Lemma 5.4 and the Lemma 5.7 proved earlier, shows that the maximum likelihood estimator $\hat{\tau}_{\epsilon}$ will lie in the bounded interval $[a, b]$ with probability tending to one as $\epsilon \rightarrow 0$ from Theorem 5.1 in Chapter 1, p.42 of Ibragimov and Khasminskii [12].

We now give a proof of Theorem 3.1.

Proof of Theorem 3.1: In view of Theorem 4.1, it follows that the family of processes $\{L_{\epsilon}(v), v \in [a, b]\}, \epsilon > 0$ on $C[a, b]$ forms a tight family. From the remarks made earlier, it follows that the finite dimensional distributions of the process $\{L_{\epsilon}(v), v \in [a, b]\}, \epsilon > 0$ are asymptotically Gaussian with the mean $-\frac{1}{2} J_{\tau}^{*2} |v|$ and the variance $J_{\tau}^{*2} |v|$ as $\epsilon \rightarrow 0$. Hence it follows that the processes $\{L_{\epsilon}(v), v \in [a, b]\}, \epsilon > 0$ on $C[a, b]$ converge weakly to the process $\{L_0(v), v \in [a, b]\}$ on $C[a, b]$ as $\epsilon \rightarrow 0$.

Let \hat{v}_ϵ denote the infimum of the points of the maxima of the process $\{L_\epsilon(v), v \in [a, b]\}$, $\epsilon > 0$ on $C[a, b]$. Let v_0 denote the location of the maxima of the process $\{L_0(v), v \in [a, b]\}$ on $C[a, b]$. The location v_0 of the maxima is unique almost surely by the fact that the Wiener process has increments which are Gaussian. Since the process $\{L_\epsilon(v), v \in [a, b]\}$, $\epsilon > 0$ on $C[a, b]$ converge weakly to the process $\{L_0(v), v \in [a, b]\}$ on $C[a, b]$ as $\epsilon \rightarrow 0$., by the continuous mapping theorem, it follows that the distribution of \hat{v}_ϵ converges in law to the distribution of v_0 by the continuous mapping theorem (cf. Billingsley [2]). Lemma 5.7 implies that the random variable $\hat{v}_\epsilon = \epsilon^{-2}[\hat{\tau}_\epsilon - \tau_0] \in [a, b]$ with probability tending to one as $\epsilon \rightarrow 0$. Let τ be the true change point. Following the discussion given above, it follows that the random variable

$$\hat{v}_\epsilon = \epsilon^{-2}[\hat{\tau}_\epsilon - \tau]$$

converges in law to the distribution of the random variable v_0 , the location of the maximum of the process $\{L_0(v), -\infty < v < \infty\}$, as $\epsilon \rightarrow 0$.

Remarks : We have assumed that the Hurst index H of the driving force for the fractional diffusion process is known through out the earlier discussions. It would be interesting to find out the asymptotic behaviour of a suitably transformed maximum likelihood estimator $\hat{\tau}_\epsilon$ when H is unknown by using an estimator \hat{H}_n of H and studying the corresponding plug-in-estimator.

Example : Suppose the process $\{X_t, 0 \leq t \leq T\}$ satisfies the stochastic differential system

$$\begin{aligned} dX(t) &= gdt + \epsilon dW_t^H, 0 \leq t \leq \tau \\ dX(t) &= hdt + \epsilon dW_t^H, \tau < t \leq T \end{aligned}$$

where g and h are constants with $g \neq h$. . This is the problem of estimation of the change point for a fractional Brownian motion with a linear shift. It can be seen that the function $A(t, x)$ does not depend on x in this example and the conditions (A.1) – (A.5) hold. Let $\hat{\tau}_\epsilon$ be the maximum likelihood estimator. An application of Theorem 3.1 implies that the random variable $\epsilon^{-2}(\hat{\tau}_\epsilon - \tau)$ has a limiting distribution as $\epsilon \rightarrow 0$.

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