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Characterization of a Multivariate Normal Distribution from Samples of Random Size

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Abstract: We obtain two characterizations of a multivariate normal distribution from samples of random size.

Key words: Characterization; Multivariate normal distribution; Samples of random size.

1 Introduction

Let \( X_i, 1 \leq i \leq N \) and \( Y_j, 1 \leq j \leq N \) be two independent samples of independent identically distributed \( k \)-dimensional random vectors with \( X_i \) distributed as \( F \) and \( Y_j \) distributed as \( G \) where \( N \) is a discrete integer valued random variable independent of \( X_i, 1 \leq i \leq N \) and \( Y_j, 1 \leq j \leq N \). Let

\[
W = \sum_{j=1}^{N} [(a - X_j)'\Sigma^{-1}(a - X_j) + (b - Y_j)'\Sigma^{-1}(b - Y_j)]
\]

where \( \Sigma \) is a known positive definite matrix for vectors \( a \) and \( b \) in \( \mathbb{R}^k \). Suppose that \( E[e^{-\frac{1}{2}W}] = J(a, b) < \infty \). We prove that the function \( J(a, b) \) is a measurable function of the function \( a'\Sigma^{-1}a + b'\Sigma^{-1}b \) if and only if the distributions \( F \) and \( G \) are multivariate normal with mean zero vector and common covariance matrix \( \sigma^2\Sigma \) for some constant \( \sigma^2 > 1 \). This result generalizes a similar result in the univariate case by Kotlarski and Cook (1977). Characterization problems of similar nature for identifiability in stochastic models are discussed in Prakasa Rao (1992).

2 Characterizations

We now state and prove the main results.

Theorem 2.1: Suppose that the function \( J(a, b) = E[e^{-\frac{1}{2}W}] < \infty \) for all vectors \( a \) and \( b \)
in $\mathbb{R}^k$. Then the function $J(a, b)$ is a measurable function of the function $a'\Sigma^{-1}a + b'\Sigma^{-1}b$ for $a, b \in \mathbb{R}^k$ if and only if the distributions $F$ and $G$ are multivariate normal with mean zero vector and common covariance matrix $\sigma^2\Sigma$ for some positive constant $\sigma^2 > 1$.

**Proof:** It is clear that

$$E[e^{-\frac{1}{2}W}] = \sum_{n=1}^{\infty} E[e^{-\frac{1}{2}W | N = n}]P(N = n)$$

$$= \sum_{n=1}^{\infty} (E[\exp(-\frac{1}{2}(a - X_j)'\Sigma^{-1}(a - X_j))]E[\exp(-\frac{1}{2}(b - Y_j)'\Sigma^{-1}(b - Y_j))])^n P(N = n).$$

The last inequality follows from the assumption that $X_i, 1 \leq i \leq N$ and $Y_j, 1 \leq j \leq N$ are two independent samples of independent identically distributed $k$-dimensional random vectors independent of the random variable $N$. Let

$$\alpha(a) = E[\exp(-\frac{1}{2}(a - X_j)'\Sigma^{-1}(a - X_j))]$$

and

$$\beta(b) = E[\exp(-\frac{1}{2}(b - Y_j)'\Sigma^{-1}(b - Y_j))].$$

Then, it follows that,

$$E[e^{-\frac{1}{2}W}] = Q(\alpha(a)\beta(b))$$

where

$$Q(x) = \sum_{n=1}^{\infty} x^n P(N = n)0 \leq x \leq 1.$$

Note that the function $Q(.)$ is a strictly increasing continuous function on the interval $[0, 1]$. Hence its inverse is well defined. Suppose that the function $E[e^{-\frac{1}{2}W}]$ is a measurable function of the function $a'\Sigma^{-1}a + b'\Sigma^{-1}b$. Then there exists a measurable real-valued function $\psi(.)$ such that

$$\psi(a'\Sigma^{-1}a + b'\Sigma^{-1}b) = Q(\alpha(a)\beta(b))$$

or equivalently

$$\alpha(a)\beta(b) = \gamma(a'\Sigma^{-1}a + b'\Sigma^{-1}b)$$

where $\gamma = Q^{-1}\alpha\psi$ for all $a, b \in \mathbb{R}^k$. It is easy to see that $\alpha(0) \neq 0$ and $\beta(0) \neq 0$ for $a = 0$ and $b = 0$. Substituting $a = 0$ and $b = 0$ alternately, we obtain that

$$\gamma(a'\Sigma^{-1}a)\gamma(b'\Sigma^{-1}b) = \alpha(0)\beta(0)\gamma(a'\Sigma^{-1}a + b'\Sigma^{-1}b)$$

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for all \(a, b \in \mathbb{R}^k\). Let

\[
\theta(t) = \frac{\gamma(t)}{\alpha(0)\beta(0)}, t \geq 0.
\]

Note that the function \(\theta(.)\) is measurable and the equation (2.3) implies that

\[
\theta(a'\Sigma^{-1}a)\theta(b'\Sigma^{-1}b) = \theta(a'\Sigma^{-1}a + b'\Sigma^{-1}b)
\]

for all \(a, b \in \mathbb{R}^k\). Hence the function \(\theta(.)\) is a measurable function such that

\[
\theta(t)\theta(s) = \theta(t+s)
\]

for all \(t, s \geq 0\) since \(\Sigma^{-1}\) is a positive definite matrix. Therefore

\[
\theta(t) = e^{ct}, t \geq 0
\]

for some constant \(c\). Hence

\[
\gamma(t) = e^{ct}\alpha(0)\beta(0), t \geq 0.
\]

Therefore, for any \(a \in \mathbb{R}^k\),

\[
\gamma(a'\Sigma^{-1}a) = e^{a'\Sigma^{-1}a}\alpha(0)\beta(0), a \in \mathbb{R}^k.
\]

Note that

\[
\gamma(a'\Sigma^{-1}a) = \alpha(a)\beta(0), a \in \mathbb{R}^k
\]

from (2.2). Combining the equations (2.8) and (2.9) and noting that \(\beta(0) \neq 0\), it follows that

\[
e^{a'\Sigma^{-1}a}\alpha(0) = \alpha(a) = \int_{\mathbb{R}^k} \exp[-\frac{1}{2}(a-x)'\Sigma^{-1}(a-x)]F(dx).
\]

This in turn gives the relation

\[
\frac{\alpha(0)}{(2\pi)^k/2|\Sigma|^{1/2}}e^{a'\Sigma^{-1}a} = \int_{\mathbb{R}^k} \frac{1}{(2\pi)^k/2|\Sigma|^{1/2}} \exp[-\frac{1}{2}(a-x)'\Sigma^{-1}(a-x)]F(dx)
\]

for all \(a \in \mathbb{R}^k\). The expression on the right side of the equation (2.11) is the convolution of a multivariate normal density function with the distribution \(F\). Hence the expression on the left side of the equation (2.11) also has to be a probability density function which implies that the constant \(c = -\frac{1}{2\sigma^2}\) for some \(\sigma^2 > 0\) and \(\alpha(0)\) is a suitable normalizing constant.

The characteristic functions of the probability densities on both sides of the equation (2.11), then, should satisfy the relation

\[
\exp[-\frac{1}{2}(t'\Sigma t)\sigma^2] = \exp[-\frac{1}{2}t'\Sigma t] \phi_X(t), t \in \mathbb{R}^k
\]
where $\phi_X$ is the characteristic function of the random vector $X$. Hence

$$\phi_X(t) = \exp[-\frac{1}{2}(t'(\sigma^2\Sigma - \Sigma)t)s^2], \, t \in \mathbb{R}^k.$$  \hspace{1cm} (2. 13)

Since $\phi_X$ is the characteristic function of the random vector $X$, it follows that $\sigma^2 > 1$ and the random vector $X$ has the multivariate normal distribution with the mean vector zero and the covariance matrix $(\sigma^2 - 1)\Sigma$. Similar arguments prove that the random vector $Y$ is also multivariate normal with mean vector zero and the covariance matrix $(\sigma^2 - 1)\Sigma$.

The converse part of the result stated in the theorem can be easily verified.

Suppose $f$ and $g$ are probability density functions on $\mathbb{R}^k$. Let

$$Z = \Pi_{j=1}^N f(a - X_j)g(b - Y_j), \, a, b \in \mathbb{R}^k.$$  

**Theorem 2.2**: Suppose that the function $H(a, b) = E[Z] < \infty, a, b \in \mathbb{R}^k$. Then the function $H(a, b)$ is a measurable function of the function $\alpha(a)$ and $\beta(b)$ if and only if the distributions $F$ and $G$ are multivariate normal with mean vectors $\mu_F$ and $\mu_G$ and covariance matrices $\Sigma_F$ and $\Sigma_G$ respectively and the probability density functions $f$ and $g$ are multivariate normal probability density functions with mean vectors $\mu_f$ and $\mu_g$ and the covariance matrices $\Sigma_f$ and $\Sigma_g$ respectively with

$$\mu_f + \mu_g = 0$$

and

$$\Sigma_f + \Sigma_g = \Sigma_f + \Sigma_g = \sigma^2 \Sigma$$

for some $\sigma^2 > 0$.

**Proof**: Let $\alpha(a) = E[f(a - X)]$ and $\beta(b) = E[g(b - Y)], a, b \in \mathbb{R}^k$. It is easy to check that

$$E[Z] = \sum_{n=1}^{\infty} E[f(a - X)]E[g(b - Y)])n P(N = n)$$

$$= Q(\alpha(a)\beta(b)) \quad \text{(say)}.$$  \hspace{1cm} (2. 14)

Suppose that $E(Z) = \psi(a'\Sigma^{-1}a + b'\Sigma^{-1}b)$ for some function $\psi(.)$ Then

$$Q(\alpha(a)\beta(b)) = \psi(a'\Sigma^{-1}a + b'\Sigma^{-1}b), \, a, b \in \mathbb{R}^k.$$
This relation is similar to that in equation (2.1). Arguments similar to those given earlier show that there exists a constant $c$ such that

\begin{equation}
\alpha(0) \exp[c \ a' \Sigma^{-1} \ a] = \int_{\mathbb{R}^k} f(a - x) \ F(dx), \ a, b \in \mathbb{R}^k. \tag{2.15}
\end{equation}

Note that the expression on the right side of the equation (2.15) is the convolution of the probability density function $f$ with the distribution function $F$. Hence the function on the left side of the equation (2.15) has to be a probability density function which implies that $c = -\frac{1}{2\sigma^2}$ for some $\sigma^2 > 0$ and $\alpha(0)$ is a suitable normalizing constant for the multivariate normal density function with mean vector zero and the covariance matrix $\sigma^2 \Sigma$. An application of the Cramer’s theorem in $R^k$ proves that $f$ and $F$ are multivariate normal probability density function and distribution function respectively with

$$\mu_f + \mu_F = 0$$

and

$$\Sigma_f + \Sigma_F = \sigma^2 \Sigma.$$

Similar arguments show that $g$ and $G$ are also multivariate normal probability density function and distribution function respectively with

$$\mu_g + \mu_G = 0$$

and

$$\Sigma_f + \Sigma_G = \sigma^2 \Sigma.$$

The converse part of the result in Theorem 2.2 can be established easily. We omit the details.

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