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FILTERED FRACTIONAL POISSON PROCESSES

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Abstract : We introduce a class of processes termed as filtered fractional Poisson processes (FFPP) and study their properties and give some application of these to stochastic models. In addition, we study filtered fractional Levy processes (FFLP) as a generalization of these models.

Key words and Phrases : Filtered fractional Poisson process, Filtered fractal time Poisson process; Filtered fractional Levy process; Mittag-Leffler distribution.

Mathematics Subject Classification : Primary 60G22, Secondary 33E12.

1 Introduction

Laskin (2003) investigated a processs called a fractional non-Markov Poisson stochastic process based on fractional generalization of the Kolomogorov-Feller equation. Matzler and Klafter (2000), Saichev and Zaslavsky (1997) and Zaslavsky (2002) observed that the main experimentally observed features of anomalous kinetic phenomena in complex systems are non-exponential time and non-Gaussian space patterns. The non-exponential evolution is caused by the long-run memory effects in complex systems. One of the important features of statistical inference of a counting process is the analysis of inter-arrival times. It is known that the Poisson model leads to the exponential distribution of the inter-arrival times. The fractional Poisson process (FPP) was introduced and investigated by Repin and Saichev (2000) and Laskin (2003) to model counting processes with inter-arrival times which are possibly non-exponential. Mainardi et al. (2004, 2007) and more recently by Meerschaert et al. (2011) investigated a fractional generalization of the Poisson process and renewal processes with non-exponential inter-arrival times. Laskin (2003) obtained the probability of n arrivals by time t for a fractional steam of events and the probability density function of the inter-arrival times of the fractional Poisson process. The fractional Poisson process captures long-memory effect resulting in non-exponential waiting time distribution observed empirically in complex systems. In contrast with Poisson process with a rate parameter λ, the fractional Poisson process has two parameters $\beta \in (0, 1]$ and $\lambda > 0$. If $\beta = 1$, then the FPP reduces to the Poisson process with parameter λ. Fractional Poisson process is a renewal process with waiting times following the Mittag-Leffler distribution. Mittag-Leffeler distribution is a generalization of the exponential distribution and has been introduced by Pillai (1990). Parameter estimation for fractional Poisson process is studied in Cahoy et al. (2010). Large deviations for fractional Poisson processes are discussed in Beghin and Macci (2012). Our main contribution is to introduce new stochastic models based on processes termed as the filtered fractional Poisson processes(FFPP) and the filtered fractional Levy processes (FFLP) following the ideas of filtered Poisson processes due to Parzen (1962). We will give several examples motivating this class of processes and study their properties.

2 Fractional Poisson Process

The *filtered fractional Poisson process* (FPP) is a generalization of the Poisson process. It is a renewal process with independent and identically distributed (i.i.d.) waiting times $J_n, n \geq 1$, such that

(2. 1) $$P(J_1 > t) = E_\beta(-\lambda t^\beta), t \geq 0$$

for some $0 < \beta \leq 1$, where

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}$$

denotes the Mittag-Leffler function (cf. Laskin (2003)). If $\beta = 1$, then the waiting times $J_n, n \geq 1$, are i.i.d. exponential with rate λ since $E_1(z) = e^z$. Let $T_n = J_1 + \ldots + J_n$ be the time of the n-th jump of the FPP. Then the FPP

$$N_\beta(t) = \max\{n \geq 0 : T_n \leq t\}$$

with index β and rate λ is a renewal process with the Mittag-Leffler inter-arrival times. If $\beta = 1$, then the FPP reduces to the Poisson process with rate or intensity parameter λ. Let $\{D(t), t \geq 0\}$ be a right-continuous strictly increasing process with left-limits such that

$$E[e^{-sD(t)}] = e^{-ts^\beta}, s > 0$$

for some $0 < \beta < 1$. Let

(2. 2) $$R(t) = \inf\{r > 0 : D(r) > t\}.$$

Let $N_1(t)$ be a Poisson process with rate parameter λ. We will call the process $\{N_1(R(t)), t \geq 0\}$ as the *fractal time Poisson process* (FTPP) following Meerschaert et al. (2011). It is a

Poisson process with rate λ time-changed via the process $\{R(t), t \geq 0\}$. Meerschaert et al. (2011) proved that, for $0 < \beta < 1$, the processes $\{N_1(R(t)), t \geq 0\}$ and $\{N_\beta(t)), t \geq 0\}$ have the same probabilistic structure. Hence the waiting times of the FTPP are also i.i.d. Mittag-Leffler as described earlier. From the results in Laskin (2003) and Cahoy et al.(2010), it follows that

$$P(J_1 > t) = E[\exp(-\lambda t^\beta / D(1)^\beta)] = E[\exp(-\lambda R(t))]$$

and $R(t) = [\frac{t}{D(1)}]^\beta$. Results in Bingham (1971) show that the moment generating function of the hitting time $R(t)$ is Mittag-Leffler as given in equation (2.1). The results in Meerschaert et al. (2011) uses the fact that if $D(t)$ is a β-stable subordinator and T_1 is an exponential random variable, then $D(T_1)$ has the Mittag-Leffler distribution. The random variable $D(t)$ and $t^{1/\beta}D(1)$ are identically distributed. If W_1 is exponential with parameter λ, then Pillai (1990) proved that the random variable $W_1^{1/\beta}D(1)$ has the Mittag-Leffler distribution. The Mittag-Leffler distribution is also known as the positive Linnik law (cf. Huillet (2000)).

Let

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma + \beta k)}.$$

For completion, we now list some properties of the fractional Poisson process with parameter β and λ as compared to a Poisson process with parameter λ following Laskin (2003) and Cahoy and Polito (2013). For proofs, see Laskin (2003).

(i) The tail distribution of the waiting time J_1 for the FPP with parameters β and λ is given by

$$P(J_1 > t) = E_\beta(-\lambda t^\beta)$$

and, for the Poisson process with parameter λ , it is

$$P(J_1 > t) = e^{-\lambda t}.$$

(ii) The probability density function of the random variable J_1 for the FPP with parameters β and $\lambda$is given by

$$f(t) = \lambda t^{\beta-1} E_{\beta,\beta}(-\lambda t^\beta), t > 0$$

and, for the Poisson process with parameter λ , it is

$$f(t) = \lambda e^{-\lambda t}, t > 0.$$

(iii) The probability function for the FPP with parameters β and λ is given by

$$P(N_\beta(t) = n) = \frac{(\lambda t^\beta)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\lambda t^\beta)^k}{\Gamma(\beta(k+n)+1))}$$

and, for the Poisson process with parameter λ , it is

$$P(N_1(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

(iv) The mean of the FPP with with parameters β and λ is

$$\frac{\lambda t^\beta}{\Gamma(\beta+1)}$$

and, for the Poisson process with parameter λ , it is

$$\lambda t.$$

(v) The variance of the FPP with with parameters β and λ is

$$\frac{\lambda t^\beta}{\Gamma(\beta+1)} + (\lambda t^\beta)^2 [\frac{1}{\beta\Gamma(2\beta)} - \frac{1}{\Gamma^2(\beta+1)}$$

and, for the Poisson process with parameter λ , it is

$$\lambda t.$$

(vi) The k-th moment of the FPP with with parameters β and λ is

$$(-1)^k \frac{\partial^k}{\partial s^k} E_\beta[\lambda(e^{-s}-1)t^\beta]|_{s=0}$$

and, for the Poisson process with parameter λ , it is

$$(-1)^k \frac{\partial^k}{\partial s^k} \exp[\lambda(e^{-s}-1)t]|_{s=0}.$$

3 Filtered Fractional Poisson Process (FFPP)

Definition : A stochastic process $\{X(t), t \geq 0\}$ is said to be a *filtered fractional Poisson Process* (FFPP) with the index $\beta, 0 < \beta \leq 1$ and the rate $\lambda > 0$, if it can be represented in the form

$$X(t) = \sum_{m=1}^{N_\beta(t)} w(t, \tau_m, Y_m), t \geq 0 \tag{3. 1}$$

where (i) the process $\{N_\beta(t), t \geq 0\}$ is a fractional Poisson process with the index β, and the rate λ (ii) the sequence $\{Y_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables Y, independent of the process $\{N_\beta(t), t \geq 0\}$ and (iii) the function $w(t, \tau, y)$ is a function called the *response function.*

If τ_m represents the time at which an event occurred according to the fractional Poisson process $\{N_\beta(t), t \geq 0\}$, then the random variable Y_m represents the amplitude of the signal associated with the event and $w(t, \tau_m, y)$ is the value at time t of a signal of magnitude y originating at time τ_m and $X(t)$ represents the value at time t of the signals arising from the events occurring up to time t.

We now present an example od a phenomenon where FFPP can be used for its stochastic modeling. This example, in the context of filtered Poisson process, is discussed in Parzen (1962).

Example 3.1: Consider a telephone exchange system with an infinite number of channels. Each call gives rise to a conversation on one of the free channels available. Suppose the subscribers make calls at times $\tau_1, \tau_2, \ldots$ where $0 < \tau_1 < \tau_2 < \ldots$ and the arrival of calls follow a FFPP with the index β and the rate λ. The holding time, that is the duration of the conversation, of the subscriber calling at time τ_n, is denoted by Y_n. We assume that the sequence $\{Y_n, n \geq 1\}$ are i.i.d. random variables. Let $X(t)$ denote the number of channels busy at time t. Note that $X(t)$ is the number of instants τ_n for which $\tau_n \leq t \leq \tau_n + Y_n$. This in turn can be represented in the form

$$X(t) = \sum_{n=1}^{N_\beta(t)} w_0(t - \tau_n, Y_n)$$

where

$$\begin{aligned} w_0(s, y) &= 1 \text{ if } 0 \leq s \leq y \\ &= 0 \text{ if } s < 0 \text{ or } s > y \end{aligned}$$

and $N_\beta(t)$ denotes the number of calls in the interval $[0, t]$.

The example described above gives a stochastic model FFPP for the number of busy channels in a telephonic system. A similar FFPP stochastic model can be considered for modeling (i) the number of busy servers in an infinite-server queue, (ii) the number of claims in force on a workman' compensation insurance policy, or (iii) the number of pulses locking a paralyzable counter. For such models built on a Poisson model, see Parzen (1962). The function

$$w(t, \tau, y) = w_0(t - \tau, y)$$

where the function $w_0(s,y)$, as defined above, is the response function for the Example 3.1 of the FFPP described above. Here the the effect at time t of a signal occurring at time τ depends only on the difference $t-\tau$. One can consider other examples of response functions which can be used for building stochastic models of the FFPP. These are functions of the type

$$\begin{aligned} w_0(s,y) &= 1 \text{ if } 0<s<y \\ &= 0 \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} w_0(s,y) &= y-s \text{ if } 0<s<y \\ &= 0 \text{ otherwise.} \end{aligned}$$

These functions give rise to stochastic models of FFPP useful in business models. Functions of the type

$$w_0(s,y) = y\, w_1(s) \tag{3. 2}$$

where $w_1(s)$ is a suitable function with $w_1(s)=0$ for $s<0$ are used as response functions for models in shot noise. The choice of the function

$$\begin{aligned} w_1(s) &= 1 \text{ if } s\geq 0 \\ &= 0 \text{ if } s<0 \end{aligned}$$

leads to compound fractional Poisson process.

4 Moments of FFPP

Suppose the response function $w(t,\tau,y)$ has the property that $w(t,\tau,y)=0$ if $t<\tau$. In other words, a signal occurring at time τ has no influence on happenings at an earlier time τ. We will now evaluate the characteristic function of the random vector $(X(t_1),X(t_2))$ and obtain formulae for the computation of moments and covariance function of the of process $\{X(t),t\geq 0\}$ whenever they exist. Without loss of generality, suppose that $0\leq t_1<t_2<\infty$. Note that the process $\{X(t),t\geq 0\}$ can be represented in the form

$$X(t)=\sum_{m=1}^{\infty} w(t,\tau_m,Y_m)$$

and hence, for any $-\infty < u_1, u_2 < \infty$,

$$u_1\, X(t_1) + u_2\, X(t_2) = \sum_{m=1}^{N_\beta(t_2)} g(\tau_m, Y_m)$$

where

$$g(\tau, y) = u_1\, w(t_1, \tau, y) + u_2\, w(t_2, \tau, y).$$

Note that

$$\Phi(u_1, u_2) = E[e^{iZ}]$$

is the characteristic function of the bivariate random vector $(X(t_1), X(t_2))$ where

$$Z = \sum_{m=1}^{N_\beta(t_2)} g(\tau_m, Y_m).$$

Observe that

$$\begin{aligned} E[e^{iZ}] &= \sum_{n=0}^{\infty} E[e^{iZ} | N_\beta(t_2) - N_\beta(0) = n] P[N_\beta(t_2) - N_\beta(0) = n] \\ &= \sum_{n=0}^{\infty} E[e^{iZ} | N_1(R(t_2)) - N_1(R(0)) = n] P[N_1(R(t_2)) - N_1(R(0)) = n]. \end{aligned}$$

From Theorem 2.2 of Meerschaert et al. (2011), it follows that, for any $0 < \beta < 1$, the fractal time Poisson process (FTPP) $\{N_1(R(t))), t \geq 0\}$ is a FPP and the waiting times between the jumps of the FTPP are i.i.d. Mittag-Leffler. Let W_n be an i.i.d. sequence with $P(W_n > t) = e^{-\lambda t}$ and $V_n = W_1 + \ldots + W_n$. Let $\tau_n = \sup\{t > 0 : N_1(R(t)) < n\}$ denote the jump times of the FTPP. From the fact that $[N_1(t) < n] = [V_n > t]$, it follows that

$$\tau_n = \sup\{t > 0 : R(t) < V_n\}.$$

Let $X_1 = \tau_1$ and $X_n = \tau_n - \tau_{n-1}$ for $n \geq 2$. Then the sequence $\{X_n, n \geq 1\}$ are the waiting times between the jumps of the FTPP and they are i.i.d. Mittag-Leffler. Furthermore

$$E[e^{-s\tau_1}] = \frac{\lambda}{\lambda + s^\beta}, s \geq 0$$

and

$$E[e^{-s_1\tau_1 - s_2\tau_2}] = \frac{\lambda}{\lambda + (s_1 + s_2)^\beta} \frac{\lambda}{\lambda + s_2^\beta}, s_1 \geq 0, s_2 \geq 0$$

from the results in Meerschaert et al. (2011).

Suppose that n events of the FPP $\{N_\beta(t), t \geq 0\}$ occur at epochs $0 < t_1 < \ldots < t_n \leq t$ during $(0,t]$ with $P(N_\beta(0) = k) = 1$ if $k = 0$ and $P(N_\beta(0) = k) = 0$ if $k \geq 1$. The joint probability density function of $(\tau_1, \tau_2, \ldots, \tau_n)$ and $N_\beta(t)$ is given by

$$P(\tau_1 \in (t_1, t_1 + dt_1), \ldots, \tau_n \in (t_n, t_n + dt_n) \text{ and no event occurs in } (t_n + dt_n, t]))$$
$$= \{\Pi_{i=1}^n [\lambda(t_i - t_{i-1})^{\beta-1} E_{\beta,\beta}(-\lambda(t_i - t_{i-1})^\beta)]\} E_\beta(-\lambda(t - t_n)^\beta)$$
$$= \lambda^n [\Pi_{i=1}^n (t_i - t_{i-1})^{\beta-1}][\Pi_{i=1}^n E_{\beta,\beta}(-\lambda(t_i - t_{i-1})^\beta)] E_\beta(-\lambda(t - t_n)^\beta)$$

where

$$E_\beta(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1 + \beta k)}$$

as defined earlier and

$$E_{\beta,\gamma}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\gamma + \beta k)}.$$

This follows from the fact that the inter-arrival density for the fractional Poisson process with parameters β and λ is given by

$$\begin{aligned} f_\beta(t) &= \lambda t^{\beta-1} E_{\beta,\beta}(-\lambda t^\beta), t \geq 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

from the results in Laskin (2003). Furthermore

$$P(N_\beta(t) = n) = \frac{(\lambda t^\beta)^n}{n!} \sum_{k=0}^\infty \frac{(k+n)!}{k!} \frac{(-\lambda t^\beta)^k}{\Gamma(\beta(k+n)+1)} \tag{4. 1}$$

(cf. Laskin (2003), Eqn (25); Cahoy and Polito (2013)). Hence the conditional density function of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ given that $N_\beta(t) = n$ is given by

$$h_{\lambda,\beta}(t_1, \ldots, t_n) =$$
$$\frac{n!}{t^{\beta n}} \frac{[\Pi_{i=1}^n (t_i - t_{i-1})^{\beta-1}][\Pi_{i=1}^n E_{\beta,\beta}(-\lambda(t_i - t_{i-1})^\beta)] E_\beta(-\lambda(t - t_n)^\beta)}{\sum_{k=0}^\infty \frac{(k+n)!}{k!} \frac{(-\lambda t^\beta)^k}{\Gamma(\beta(k+n)+1)}} dt_1 \ldots dt_n$$

for $0 < t_1 < \ldots < t_n < t$ and zero otherwise. If $\beta = 1$, then the the FPP reduces to the Poisson process with intensity parameter λ and the density function given above reduces to the density function of the order statistics of a random sample of size n from the Uniform distribution on the interval $[0, t]$.

We shall now evaluate the characteristic function of the random variable Z following the technique in Parzen (1962), p.155. Observe that $P(N_\beta(0) = 0) = 1)$, and

$$E[e^{iZ}] = \sum_{n=0}^{\infty} E[e^{iZ}|N_\beta(t_2) = n]P[N_\beta(t_2) = n].$$

Let A be the event that $N_\beta(t_2) = n$ and

$$\eta_n(u_1, u_2) = E_Y[\exp\{i \sum_{m=1}^{n} g(\tau_m, Y_m)\}|A].$$

For real numbers $s_1, \ldots, s_n$ satisfying $0 \leq s_1 < s_2 < .... < s_n \leq t_2$, define

$$\phi(s_1, ..., s_n) = E_Y[\exp\{i \sum_{m=1}^{n} g(s_m, Y_m)\}|A, \tau_1 = s_1, ..., \tau_n = s_n].$$

Then

$$\phi(s_1, ..., s_n) = \Pi_{m=1}^{n} E_Y[\exp\{ig(s_m, Y)\}].$$

Hence

$$\begin{aligned} \eta_n(u_1, u_2) &= \int_0^{t_2} ds_1 \int_{s_1}^{t_2} ds_2 \ldots \int_{s_{n-1}}^{t_2} \phi(s_1, \ldots, s_n) h_{\lambda,\beta}(s_1, \ldots, s_n) ds_n \\ &= \int_0^{t_2} ds_1 \int_{s_1}^{t_2} ds_2 \ldots \int_{s_{n-1}}^{t_2} \Pi_{m=1}^{n} E_Y[\exp\{ig(s_m, Y)\}] h_{\lambda,\beta}(s_1, \ldots, s_n) ds_n. \end{aligned}$$

Therefore

$$\begin{aligned} E[e^{iZ}] &= \sum_{n=0}^{\infty} \eta_n(u_1, u_2) P(N_\beta(t_2) = n) \\ &= \sum_{n=0}^{\infty} \eta_n(u_1, u_2) \frac{(\lambda t_2^\beta)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\lambda t_2^\beta)^k}{\Gamma(\beta(k+n)+1))} \end{aligned}$$

(see Laskin (2003), Eqn. (54) for computation of the moment generating function of Z). As a consequence of this representation of the characteristic function of the random vector $(X(t_1), X(t_2))$, one can compute the moments of the process $X(t)$ using the formulae

$$i\ E[X(t_1)] = \frac{d}{du_1} \log \Phi(u_1, 0)|_{u_1=0},$$

$$i^2\ Var[X(t_1)] = \frac{d}{du_1^2} \log \Phi(u_1, 0)|_{u_1=0},$$

and

$$i\, Covar[X(t_1), X(t_2)] = \frac{\partial^2}{\partial u_1 \partial u_2} \log \Phi(u_1, u_2)|_{u_1=0, u_2=0}$$

following the standard results connecting characteristic functions of random vectors and joint moments of random vectors (cf. Roussas (1973), p. 111)). Our approach for computation of mean and other moments is analogous to that in Parzen (1962) based on characteristic functions of the random vectors which exist always for any probability distribution where as Laskin (2003) approached the problem through moment generating functions whose existence has to be established in any given problem. Laskin (2003) computes the moment generating function of a FCPP and derived its mean function. We consider more general FFPP and derive formulae for its mean and covariance function. Specific computation depends on the choice of the response function $w(t, \tau, y)$.

5 Applications

(i) Let $X(t)$ be the number of busy channels in an exchange with an infinite number of channels or the number of busy servers in an infinite-server queue or the number of workmen who are drawing compensation insurance at time t dealing with the number of claims in force on a workman's compensation insurance. We assume that the arrival of calls or of customers are events of fractional Poisson type with parameters β and λ and that the service or holding times are independent and identically distributed with a distribution with non-negative support. Here the process $\{X(t), t \geq 0\}$ is a filtered fractional Poisson process and its moments can be obtained by applying the results described in the previous section.

(ii) A stochastic process $\{X(t), t \in R\}$ is said to be a *fractional shot noise process* if it can be represented as the super position of impulses occurring at random times $\ldots, \tau_{-1}, \tau_0, \tau_1, \ldots$. All impulses are assumed to have the same shape $w(s)$ so that

$$X(t) = \sum_{m=-\infty}^{\infty} w(t - \tau_m).$$

As a more general case, the impulse shapes may be randomly chosen from a family of shapes $w(s, y)$ indexed by a parameter y. At each time τ_m, the parameter y is chosen as the observation of a random variable Y_m and the process $X(t)$ is defined to be the superposition

$$X(t) = \sum_{m=-\infty}^{\infty} w(t - \tau_m, Y_m).$$

The times $\{\tau_m, -\infty < m < \infty\}$ are assumed to occur according to a fractional Poisson process with parameters λ and β and the random variables $\{Y_m\}$ are assumed to be i.i.d. random variables. One can derive the mean, the variance and the covariance of the process $\{X(t), t \geq 0\}$ by following the methods described earlier. These results generalize the Campbell's theorem on the superposition of random impulses when the arrivals form a Poisson process (cf. Rice (1944)) to the fractional Poisson process.

(iii) Consider a filtered fractional Poisson process $\{X(t), t \in R\}$ given by

$$X(t) = \sum_{m=-\infty}^{\infty} w(t - \tau_m)$$

where $\{\tau_m, -\infty < m < \infty\}$ are the times of occurrence of events following a fractional Poisson process with parameters λ and β and the function

$$\begin{aligned} w(s) &= c|s|^{-\gamma} \text{ if } s \geq 0 \\ &= -w(-s) \text{ if } s < 0 \end{aligned}$$

where c and γ are positive. It is possible to give an interpretation to this filtered fractional Poisson process. Suppose the particles are distributed randomly on a line in accordance with a fractional Poisson process with parameters λ and β. Suppose the force between any two particles is one of attraction and of magnitude $cr^{-\gamma}$ where $c > 0$ and r is the distance between the particles. Then the random variable $X(t)$ represents the total force that would be exerted on a particle located at t. Since the force and acceleration are equal up to a constant factor, it is also possible to interpret the random variable $X(t)$ as the acceleration of the particle located at t. This model for acceleration may be generalized to particles distributed in space.

6 Fractional Compound Poisson Process

Laskin (2003) introduced Fractional Compound Poisson process (FCPP) and computed the moment generating function of the FCPP. A stochastic process $\{X(t), t \geq 0\}$ is said to be a *fractional compound Poisson process* (FCPP) if it can be represented as

$$X(t) = \sum_{n=1}^{N_\beta(t)} Y_n$$

where $\{N_\beta(t), t \geq 0\}$ is a FPP with parameters β and λ and $\{Y_n, n \geq 1\}$ is a family of i.i.d. random variables distributed as a random variable Y. The process $\{N_\beta(t), t \geq 0\}$ and the sequence $\{Y_n, n \geq 1\}$ are assumed to be independent.

There are several phenomena where such processes arise. An example is the total claims of policy holders of an insurance company. Suppose the policy holders of a life insurance company die at times $\tau_1, \tau_2, \ldots$ where $0 < \tau_1 < \tau_2 < \ldots$. Deaths are assumed to be events of the fractional Poisson type with parameters β and λ. The policy holder dying at time τ_n carries a policy for an amount Y_n which is paid to his beneficiary at the time of his death. The insurance company is interested in knowing $X(t)$, the total amount of claims it will have to pay in the time $[0, t]$ in order to determine how large a reserve to have on hand to meet the claims it will have to pay. Here the process $\{X(t), t \geq 0\}$ is a fractional compound Poisson process.

7 Testing for FPP

Recall that the FPP with parameters β and λ is a renewal process with i.i.d. waiting times J_n that satisfy

$$P(J_n > t) = E_\beta(-\lambda t^\beta)$$

where $0 < \beta \leq 1$ and $E_\beta(z)$ is the Mittag-Leffler function. This fact can be used to describe a method for testing the hypothesis that a sequence of events occurring in time are events of fractional Poisson type. The observed inter-arrival times $J_1, \ldots, J_n$ are assumed to be independent identically distributed observations. Using various goodness-of-fit tests, such as the Chi-square test for goodness-of-fit, one can test the hypothesis that J_1 has the Mittag-Leffler distribution.

8 Estimating the parameter λ of a fractional Poisson process

Cahoy(2013) describes a procedure for estimating the parameters of a Mittag-Leffler distribution by using the method of moments. If one observes a fractional Poisson process $N_\beta(t)$ for a fixed observation time t, then the number $N_\beta(t) = N_1(R(t))$ is fractal time Poisson process following the work in Meerschaert et al. (2011) where the process $R(t)$ is as defined by (2.2). Suppose the process is observed until a fixed number m of the events have occurred.

Let T_m be the total time required. It can be shown that

(8. 1)
$$E(e^{-sT_m}) = \frac{\lambda}{\lambda + s^\beta}, s \geq 0$$

(cf. Meerschaert et al. (2011)). Applying the method of moments,, one can estimate the parameter λ, from the equation

$$e^{-sT_m} = \frac{\lambda}{\lambda + s^\beta}$$

assuming that the parameter β is known for given s or estimate the parameters λ and β by equating the moments for two different values of s.

9 Comparing fractional Poisson processes

Let the processes $\{N_\beta(t), t \geq 0\}$ and $\{N'_\beta(t), t \geq 0\}$ be two independent fractional Poisson processes with parameters β and λ and β and λ' respectively. The problem of interest is to test whether $\lambda = \lambda'$. Let T_n be the waiting time for the n-th event under the process N and T'_m be the waiting time for the m-th event under the process N'. It would be interesting to find the distribution of the ratio T_n/T'_n under the hypothesis $\lambda = \lambda'$ and this can be used for testing the hypothesis $\lambda = \lambda'$.

10 Testing whether the events are of fractional Poisson type

Suppose that a stochastic process is observed until n events have occured. Let U_j denote the time at which the j-th event has occurred. Following Meerschaert et al. (2011), we have seen earlier that if a process is a fractional Poisson process, then the inter-arrival times $U_j - U_{j-1}, 1 \leq j \leq n$ are i.i.d. random variables and have the Mittag-Leffler distribution for some parameters β and λ. Let $S_n = \sum_{i=1}^n [U_i - U_{i-1}]$. For large values of n, the random variable S_n has approximately a normal distribution with mean and variance depending on β and λ. This can be used for testing whether the process is of Poisson type in the class of fractional Poisson processes by testing the hypothesis $\beta = 1$ against the alternate hypothesis that $\beta < 1$ and in general for testing whether the events are of fractional Poisson type among the class of all counting processes.

11 Extension of FFPP

(i) One can generalize the family of filtered fractional Poisson processes much further. Let $\{W(t,\tau), t \geq 0, \tau \geq 0\}$ be a stochastic process with characteristic function

$$\Phi_{W(t,\tau)}(u) = E[\exp\{iu\, W(t,\tau)\}].$$

Let $\{W_m, m \geq 1\}$ be a family of stochastic processes identically distributed as W. Let $\tau_1 < \ldots < \tau_m < \ldots$ be the times of occurrences of the events of fractional Poisson type with parameters λ and β. Let $N_\beta(t)$ be the number of events which have occurred in the interval $(0,t]$. Suppose the processes $\{N_\beta(t), t \geq 0\}$ and $\{W_m, m \geq 1\}$ are independent. Let

$$X(t) = \sum_{m=1}^{N_\beta(t)} W_m(t, \tau_m).$$

The process $\{X(t), t \geq 0\}$ is called a *generalized filtered fractional Poisson process* (GFFPP). One can compute the characteristic function, mean and covariance of the process X by methods similar to those discussed earlier.

As an example, we will now discuss a model for population processes with immigration. Suppose an animal of a certain species immigrates to a certain region at time τ. Then the number of descendants of this animal present in this region at time t is a random variable $W(t,\tau)$. Suppose that there are no animals of this species in this region initially but that animals of this species immigrate into the region at times $\tau_1 < \tau_2 < \ldots$ which are of fractional Poisson type with parameters λ and β. Let $W_m(t,\tau_m)$ denote the number of descendants at time t of the animal which immigrated to the region at time τ_m. Then the total number $X(t)$ of animals in the region of this species at time t is given by

$$X(t) = \sum_{m=1}^{N_\beta(t)} W_m(t, \tau_m).$$

Assuming independence of all the population processes under consideration, it can be seen that the process $\{X(t), t \geq 0\}$ is a generalized FFPP. It is easy to compute the probability generating function of the process X from the probability generating function of the process W.

(ii) Generalization of Campbell's theorem: Consider a stochastic process of the form

$$X(t) = \sum_{-\infty < \tau_n < \infty} Y_n\, w(t - \tau_m)$$

where $\{\tau_n, n \geq 1\}$ are the times of occurrences of events occurring following a fractional Poisson process with parameters λ and β and $\{Y_n, n \geq 1\}$ are i.i.d. random variables. Following the ideas explained earlier, one can compute the cumulants of the random variable $X(t)$.

12 Filtered Fractional Levy Processes

Recall that, if $\{N_1(t), t \geq 0\}$ is a Poisson process with intensity parameter λ, then $\{N_1(R(t)), t \geq 0\}$ is a FTPP where the process $\{R(t), t \geq 0\}$ is the right-continuous inverse of a process $\{D(t), t \geq 0\}$ which is a β-stable subordinator with $E[e^{-sD(t)}] = e^{-ts^\beta}$ for some $0 < \beta < 1$. Following Meerschaert et al. (2011), let $\{D_\psi(t), t \geq 0\}$ be a strictly increasing Levy process with

$$E[e^{-sD_\psi(t)}] = e^{-t\psi(s)}$$

where

$$\psi(s) = bs + \int_0^\infty (e^{-sx} - 1)\phi(dx),$$

$b \geq 0$ and $\phi(.)$ is the Levy measure of the process $\{D_\psi(t), t \geq 0\}$. Let

$$R_\psi(t) = \inf\{\tau \geq 0 : D_\psi(\tau) > t\}.$$

Then the process $\{N_1(R_\psi(t)), t \geq 0\}$ is a renewal process with i.i.d. inter-arrival times $\{J_n, n \geq 1\}$ satisfying

$$P(J_1 > t) = E[e^{-\lambda R_\psi(t)}]$$

from Theorem 4.1 in Meerschaert et al. (2011). Let $\{N_\psi(t), t \geq 0\}$ denote the renewal process such that

$$N_\psi(t) = \max\{n \geq 0 : T_n \leq t\}$$

where $T_n = J_1 + \ldots + J_n$. From Remark 4.2 of Meerschaert et al. (2011), it follows that the processes $\{N_\psi(t), t \geq 0\}$ and $\{N_1(E_\psi(t)), t \geq 0\}$ are the same. Meerschaert et al. (2011) remark that

$$P(N_\psi(t) = n) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^n}{n!} h_\psi(x, t) dx$$

where $h_\psi(t, x)$ is the probability density function of the random variable $R_\psi(t)$.

A stochastic process $\{X(t), t \geq 0\}$ is said to be a *filtered fractional Levy process* (FFLP) if it can be represented in the form

$$X(t) = \sum_{m=1}^{N_\psi(t)} w(t, \tau_m, Y_m), t \geq 0 \tag{12. 1}$$

where (i) the process $\{N_\psi(t), t \geq 0\}$ is a fractional Levy process (ii) the sequence $\{Y_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables Y, independent of the process $\{N_\psi(t), t \geq 0\}$ and (iii) the function $w(t, \tau, y)$ is a function called the *response function.*

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