CRRAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)

Research Report

Author (s): **B.L.S. Prakasa Rao**

Title of the Report: **Characterizations of probability distributions through linear forms of Q-Conditional independent random variables**

Research Report No.: **RR2016-03**

Date: **February 10, 2016**

Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org

CHARACTERIZATIONS OF PROBABILITY DISTRIBUTIONS THROUGH LINEAR FORMS OF Q-CONDITIONAL INDEPENDENT RANDOM VARIABLES

B.L.S. PRAKASA RAO
CR Rao Advanced Inst. of Mathematics, Statistics
and Computer Science, Hyderabad 500046, India

Abstract: We derive some characterizations of probability distributions based on the joint distributions of linear forms of Q-conditional independent random variables.

Mathematics Subject Classification: 60E10.

Keywords: Q-conditional independence; Q-conditional identically distributiveness; Linear forms; Characterization of probability distributions.

1 Introduction

Properties of conditional independent random variables were studied in Prakasa Rao (2009). Characterization of distributions based on functions of conditional independent random variables is investgated in Prakasa Rao (2013). Earlier discussions on the topic of conditional independence can be found in Chow and Teicher (1978), Majerak et al. (2005) and Roussas (2008). Bairamov (2011) investigated the copula representation for conditionally independent random variables. Dawid (1979, 1980) observed that many important concepts in statistics can be considered as expressions of conditional independence. It is known that conditional independence of a set of random variables does not imply independence and independence does not imply conditional independence. This can be checked from the examples given in Prakasa Rao (2009). Kagan and Székely (2016) introduced the notion of Q-independence for family of random variables. It was shown that Q-independence of a set of random variables does not imply independence and they obtained some characterizations of Gaussian distribution for linear forms of Q-independent random variables. We have obtained a characterization of probability distributions through the joint distribution of linear forms of Q-independent random variables in Prakasa Rao (2016). We now introduce the concept of a Q-class, Q-conditional independence and obtain a characterization of probability distributions through the joint distribution of linear forms of Q-conditional independent random variables. It can be shown that Q-conditional independence does not imply Q-independence and vice versa.

2 Q-conditional independence

Let $X_1, \ldots, X_n$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\phi_i(t)$ be the characteristic function of $X_i, i = 1, \ldots .n$. Following Kagan and Śzekely (2016), the collection $\{X_i, i = 1, \ldots, n\}$ is said to be *Q-independent* if the joint characteristic function of the random vector $(X_1, \ldots, X_n)$ can be represented as

$$\phi_{X_1,\ldots,X_n}(t_1, \ldots, t_n) \equiv E[\exp(it_1X_1+\ldots+it_nX_n)] = \Pi_{i=1}^n \phi_i(t_i)\ \exp\{q(t_1, \ldots, t_n)\}, t_1, \ldots, t_n \in R$$

where $q(t_1, \ldots, t_n)$ is a polynomial in $t_1, \ldots, t_n$. The random variables X_j and X_k are said to be *Q-identically distributed* if

$$\phi_j(t) = \phi_k(t) \exp\{q(t)\}, t \in R$$

where $q(.)$ is a polynomial. Two random vectors $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$ are said to be *Q-identically distributed* if the characteristic function $\phi(t_1, \ldots, t_k)$ of the random vector $(X_1, \ldots, X_k)$ differs from the the characteristic function $\psi(t_1, \ldots, t_k)$ of the random vector $(Y_1, \ldots, Y_k)$ by a factor of exponential of a polynomial in $t_1, \ldots, t_k$.

The class of all random variables which are Q-identically distributed to a random variable X is called the Q-class of the random variable X and the class of all random vectors which are Q-identically distributed to a random vector $(X_1, \ldots, X_k)$ is called the Q-class of the random vector $(X_1, \ldots, X_k)$.

It is known that two random variables could be Q-independent but not independent. For instance, if X, Y, Z are non-degenerate independent Gaussian random variables, then $X + Y$ and $X + Z$ are Q-independent but not independent.

We now extend the notion of Q-independence to Q-conditional independence.

Suppose $X_1, \ldots, X_n$ and Z are random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\phi_i(t; z)$ be the conditional characteristic function of X_i given the event $Z = z$ for $i = 1, \ldots, n$. Let $\phi_i(t_1, \ldots, t_n; z)$ be the joint conditional characteristic function of the random vector $(X_1, \ldots, X_n)$ given the event $Z = z$. The collection $\{X_i, i = 1, \ldots, n\}$ is said to be *Q-conditional independent* given Z if the joint conditional characteristic function of the random vector $(X_1, \ldots, X_n)$ given $Z = z$ can be represented as

$$\phi_{X_1,\ldots,X_n}(t_1, \ldots, t_n; z) \quad \equiv \quad E[\exp(it_1X_1 + \ldots + it_nX_n)|Z = z]]$$

$$= \Pi_{i=1}^{n}\phi_i(t_i;z)\ \exp\{q(t_1,\ldots,t_n;z)\}, t_1,\ldots,t_n \in R$$

where $q(t_1,\ldots,t_n;z)$ is a polynomial in $t_1,\ldots,t_n$ for every z in the support of the random variable Z. The random variables X_j and X_k are said to be *Q-conditional identically distributed* given Z if

$$\phi_j(t;z) = \phi_k(t;z)\exp\{q(t;z)\}, t \in R$$

where $q(.;z)$ is a polynomial for every z in the support of the random variable Z.

It can be checked that a family of random variables could be Q-conditional independent but not Q-independent and vice versa. For instance, let X, Y, W be three non-degenerate Gaussian random variables and Z be another independent random variable. Then the random variables $Z(X+Y)$ and $Z(X+W)$ are Q-conditionally independent given Z but are not Q-independent or independent.

3 Characterization through joint distribution of linear forms of independent random variables

The following result is due to Kotlarski (1967) (cf. Prakasa Rao (1992), Theorem 2.1.1).

Theorem 3.1: Suppose X_1, X_2 and X_3 are three independent real-valued random variables with non-vanishing characteristic functions. Let $Y_1 = X_1 - X_3$ and $Y_2 = X_2 - X_3$. Then the joint distribution of the random vector (Y_1, Y_2) determines the distributions of X_1, X_2 and X_3 up to a change of location.

It is possible to relax the condition on non-vanishing property of the characteristic functions of the random variables $X_i, 1 \leq i \leq 3$ in Theorem 3.1 under some additional conditions on the analyticity of the characteristic functions. For details, see Sasvári (1986) (cf. Prakasa Rao (1992), p.15).

Rao (1971) (cf. Prakasa Rao (1992), Theorem 2.1.4) obtained the following result.

Theorem 3.2: Suppose X_1, X_2 and X_3 are three independent real-valued random variables. Consider two linear forms

$$W_1 = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$W_2 = b_1X_1 + b_2X_2 + b_3X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $i \neq j$. If the characteristic functions of $X_i, 1 \leq i \leq 3$ do not vanish, then the distribution of (W_1, W_2) determines the distributions of the random variables X_1, X_2 and X_3 up to change in location.

4 Characterization through joint distribution of linear forms of conditional independent random variables

Prakasa Rao (2013) proved a variation of the following conditional version of Theorem 3.1.

Theorem 4.1: Suppose X_1, X_2 and X_3 are three conditionally independent real-valued random variables given a random variable Z. . Let $Y_1 = X_1 - X_3$ and $Y_2 = X_2 - X_3$. Suppose the conditional characteristic functions of the random variables X_1, X_2, X_3 given $Z = z$ do not vanish. Then the joint distribution of the random vector (Y_1, Y_2) given $Z = z$ determines the distributions of X_1, X_2 and X_3 given $Z = z$ up to a change of location depending on z.

Following the method in Rao (1971) (cf. Prakasa Rao(1992), Theorem 2.1.4) and Prakasa Rao (2013) , it is easy to prove the following result.

Theorem 4.2: Suppose X_1, X_2 and X_3 are three conditionally independent real-valued random variables given a random variable Z. Consider two linear forms

$$W_1 = a_1 X_1 + a_2 X_2 + a_3 X_3$$

and

$$W_2 = b_1 X_1 + b_2 X_2 + b_3 X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $i \neq j$. Suppose the conditional characteristic functions of X_1, X_2, X_3 given $Z = z$ do not vanish. Then the joint distribution of (W_1, W_2) given $Z = z$ determines the distribution of X_1, X_2 and X_3 given $Z = z$ up to a change of location depending on z. .

Note that Theorems 4.1 and 4.2 are not consequences of Theorems 3.1 and 3.2 respectively as conditional independence does not imply independence and vice versa. It is necessary to derive the results independently even though the proofs are analogous.

5 Characterization through joint distribution of linear forms of Q-independent random variabes

Results analogous to Theorems 3.1 and 3.2 for independent random variables have been proved for Q-independent random variables in Prakasa Rao (2016). The following theorems characterize Q-classes of a random variables.

Theorem 5.1: Let X_1, X_2 and X_3 be three Q-independent real-valued random variables. Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. If the joint characteristic function of the random vector (Z_1, Z_2) does not vanish, then the joint distribution of the random vector (Z_1, Z_2) determines the Q-classes of the random variables X_1, X_2 and X_3.

Theorem 5.2: Suppose X_1, X_2 and X_3 are three Q-independent real-valued random variables. Consider two linear forms

$$Z_1 = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$Z_2 = b_1X_1 + b_2X_2 + b_3X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $1 \leq i \neq j \leq 3$. If the joint characteristic function of (Z_1, Z_2) does not vanish, then the joint distribution of (Z_1, Z_2) determines the Q-classes of X_1, X_2 and X_3.

6 Characterization through joint distributions of linear forms of Q-conditional independent random variables

We now obtain some characterizations for probability distributions based on Q-conditional independent random variables extending the work in Prakasa Rao (2016) for Q-independent random variables.

Theorem 6.1: Let X_1, X_2 and X_3 be three Q-conditional independent random variables given a random variable Z. Let $W_1 = X_1 + X_2$ and $W_2 = X_2 + X_3$. If the conditional characteristic function of the random vector (W_1, W_2) given $Z = z$ does not vanish, then the conditional distribution of the random vector (W_1, W_2) given $Z = z$ determines the Q-classes of the random variables X_1, X_2, X_3 depending on $Z = z$ in the support of the random variable Z.

Proof: Let $\phi_{W_1,W_2}(t_1,t_2;z)$ denote the conditional characteristic function of the bivariate random vector (W_1,W_2) given $Z=z$ and $\phi_i(t;z)$ denote the characteristic function of the random variable X_i given $Z=z$ for $i=1,2,3$. Let $\phi_{X_1,X_2,X_3}(t_1,t_2,t_3;z)$ denote the conditional characteristic function of the trivariate random vector (X_1,X_2,X_3) given $Z=z$. It is easy to check that

$$\begin{aligned}\phi_{W_1,W_2}(t_1,t_2;z) &= \phi_{X_1,X_2,X_3}(t_1,t_1+t_2,t_2;z)\\ &= \phi_{X_1}(t_1;z)(t_1)\phi_{X_2}(t_1+t_2;z)\phi_{X_3}(t_2;z)\exp\{q_1(t_1,t_1+t_2,t_2;z)\}\end{aligned}$$

for some polynomial $q_1(t_1,t_2,t_3;z)$ by the Q-conditional independence of the random variables X_1,X_2,X_3 given $Z=z$. Suppose that $Y_i, i=1,2,3$ is another set of Q-conditionally independent random variables given $Z=z$ such the joint conditional distribution of the bivariate random vector (W_1,W_2) given $Z=z$ is the *same* as the joint conditional distribution of the random vector (T_1,T_2) where $T_1=Y_1+Y_2$ and $T_2=Y_2+Y_3$ given $Z=z$. Let $\psi_{Y_i}(t;z)$ denote the conditional characteristic function of the random variable Y_i given $Z=z$ for $i=1,2,3$. It is easy to check that

$$\phi_{T_1,T_2}(t_1,t_2;z)=\psi_{Y_1}(t_1;z)\psi_{Y_2}(t_1+t_2;z)\psi_{Y_3}(t_2;z)\exp\{q_2(t_1,t_1+t_2,t_2;z)\} \quad (6.\ 1)$$

for some polynomial $q_2(t_1,t_2,t_3;z)$ by the Q-conditional independence of the random variables Y_1,Y_2,Y_3 given $Z=z$. Since the joint conditional distributions of the random vectors (W_1,W_2) and (T_1,T_2) given $Z=z$ are the same and non-vanishing, by hypothesis, it follows that $\phi_{X_i}(t;z)\neq 0, i=1,2,3$ and $\psi_{Y_i}(t;z)\neq 0, i=1,2,3$ and

$$\begin{aligned}&\phi_{X_1}(t_1;z)\phi_{X_2}(t_1+t_2;z)\phi_{X_3}(t_2;z)\exp\{q_1(t_1,t_1+t_2,t_2;z)\}\\ &=\psi_{Y_1}(t_1;z)\psi_{Y_2}(t_1+t_2;z)\psi_{Y_3}(t_2;z)\exp\{q_2(t_1,t_1+t_2,t_2;z)\}\end{aligned}$$

for $t_1,t_2\in R$ and for any fixed z in the support of the random variable Z. Let $\zeta_i(t;z)=\log[\frac{\phi_{X_i}(t;z)}{\psi_{Y_i}(t;z)}], i=1,2,3; t\in R$ where $\eta_i(t;z)\equiv\log\phi_{X_i}(t;z)$ denotes the continuous branch of the logarithm of the characteristic function $\phi_{X_i}(t;z)$ with $\eta_i(0;z)=0$. The equations derived above imply that

$$\zeta_1(t_1;z)+\zeta_2(t_1+t_2;z)+\zeta_3(t_2;z)=q_3(t_1,t_1+t_2,t_2;z), t_1,t_2\in R \quad (6.\ 2)$$

where $q_3(t_1,t_1+t_2,t_2;z)=q_1(t_1,t_1+t_2,t_2;z)-q_2(t_1,t_1+t_2,t_2;z)$ is a polynomial in t_1 and t_2 for any fixed z in the support of the random variable Z. Hence

$$\zeta_2(t_1+t_2;z)=-\zeta_1(t_1;z)-\zeta_3(t_2;z)+q_3(t_1,t_1+t_2,t_2;z), t_1,t_2\in R. \quad (6.\ 3)$$

Applying Lemma 1.5.1 in Kagan et al. (1973), it follows that $\zeta_2(t;z)$ is a polynomial in t for any fixed z in the support of the random variable Z. It is easy to check that $\zeta_i(t;z)$ is a polynomial in t for any fixed z in the support of the random variable Z for $i = 1, 3$ from the relation (6.3). Hence

$$\phi_{X_j}(t;z) = \psi_{Y_j}(t;z) \exp\{q_j(t;z)\}, t \in R, j = 1, 2, 3 \tag{6. 4}$$

where $q_j(t;z)$ is a polynomial in t for any fixed z in the support of the random variable Z. In particular, it follows that X_j and Y_j are Q-conditionally identically distributed for $j = 1, 2, 3$ for any fixed z in the support of the random variable Z.

We now prove a result generalizing Theorem 5.2 for Q-conditional independent random variables.

Theorem 6.2: Suppose X_1, X_2 and X_3 are three Q-conditional independent random variables given a random variable Z . Consider two linear forms

$$W_1 = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$W_2 = b_1X_1 + b_2X_2 + b_3X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $1 \leq i \neq j \leq 3$. If the joint conditional characteristic function of (W_1, W_2) given $Z = z$ does not vanish, then the conditional distribution of (W_1, W_2) given $Z = z$ determines the Q-classes of the random variables X_1, X_2, X_3 depending on z for any fixed z in the support of the random variable Z.

Proof: Let $\phi_i(t;z)$ be the conditional characteristic function of the random variable $X_i, i = 1, 2, 3$ given $Z = z$. Since the conditional characteristic function of the bivariate random vector (W_1, W_2) does not vanish by hypothesis, it follows that $\phi_i(t;z) \neq 0$ for all $t \in R$ and for $i = 1, 2, 3$ for any fixed z in the support of the random variable Z. Let $\eta_i(t;z) = \log \phi_i(t;z)$ denote the continuous branch of the logarithm of the characteristic function $\phi_i(t;z)$ with $\eta_i(0;z) = 0$. Suppose $\psi_i(t;z), i = 1, 2, 3$ is another set of possible conditional characteristic functions for $X_i, i = 1, 2, 3$ respectively given $Z = z$ satisfying the hypothesis. Let $\zeta_i(t;z) = \log \psi_i(t;z)$ and

$$\gamma_i(t;z) = \eta_i(t;z) - \zeta_i(t;z), t \in R, i = 1, 2, 3.$$

Since the conditional characteristic functions of the bivariate random vector (W_1, W_2) are the same for the choice of $\phi_i, i = 1, 2, 3$ given $Z = z$ as well as $\psi_i, i = 1, 2, 3$ given $Z = z$ and the random variables $X_i, i = 1, 2, 3$ are Q-conditionally independent given Z , it follows that

$$\gamma_1(a_1t+b_1u;z)+\gamma_2(a_2t+b_2u;z)+\gamma_3(a_3t+b_3u;z) = q(a_1t+b_1u, a_2t+b_2u, a_3t+b_3u;z), t, u \in R$$

where $q(t_1, t_2, t_3; z)$ is a polynomial in t_1, t_2, t_3 for any fixed z in the support of the random variable Z. Since $a_i : b_i \neq a_j : b_j$ for $i \neq j, 1 \leq i, j \leq 3$ by hypothesis, the equation given above can be written in one of the following forms depending on the values of a_i and b_i : for any fixed z in the support of the random variable Z,

(i)$\gamma_1(t + c_1u; z) + \gamma_2(t + c_2u; z) + \gamma_3(t + c_3u; z) = q(t + c_1u, t + c_2u, t + c_3u; z), t, u \in R; c_1 \neq c_2 \neq c_3 \neq 0;$

(ii)$\gamma_1(t + c_1u; z) + \gamma_2(t + c_2u; z) = A(t; z) + q(t + c_1u, t + c_2u, 0; z); c_1 \neq c_2 \neq 0;$

(iii)$\gamma_1(t + c_1u; z) = A(t; z) + B(u; z) + q(t + c_1u, 0, 0; z); c_1 \neq 0$

where $A(t; z)$ is continuous in t for any fixed z in the support of the random variable Z, $B(u; z)$ is continuous in u for any fixed z in the support of the random variable Z and $q(t_1, t_2, t_3; z)$ is a polynomial in t_1, t_2, t_3 for any fixed z in the support of the random variable Z. Applying results due to Rao (1966, 1967) (cf. Prakasa Rao (1992), Lemma 2.1.2 and Lemma 2.1.3), it follows that the functions $\gamma_i(t; z), i = 1, 2, 3$ are polynomials of degree less than or equal to $\max(3, k)$ where k is the degree of the polynomial $q(t_1, t_2, t_3; z)$ for any fixed z in the support of the random variable Z. This in turn implies that

$$\phi_i(t; z) = \psi_i(t; z) \exp\{q_i(t; z)\}, i = 1, 2, 3$$

where $q_i(t; z)$ is a polynomial for $i = 1, 2, 3$ for any fixed z in the support of the random variable Z.

As was mentioned above, the results obtained in Theorems 6.1 and 6.2 are not consequences of the results in Theorems 5.1 and 5.2 even though the proofs are analogous.

The next result deals with a set of n Q-conditional independent real-valued random variables. Proof of Theorem 6.3 is similar to that of Theorem 6.1. We omit the proof.

Theorem 6.3: Let $X_i, 1 \leq i \leq n$ be n Q-conditional independent real-valued random variables given a random variable Z. . Define $W_i = X_i - X_n, 1 \leq i \leq n - 1$. Suppose the conditional characteristic function of $\mathbf{W} = (W_1, \ldots, W_{n-1})$ given $Z = z$ does not vanish for any fixed z in the support of the random variable Z. Then the joint conditional distribution of the random vector $\mathbf{W}$ given $Z = z$ determines the Q-classes of the random variables $X_1, X_2, \ldots, X_n$ given $Z = z$ depending on z for any fixed z in the support of the random variable Z.

Remarks:(1) Results stated in Theorems 5.1 and 5.2 will continue to hold for the Q-class of the random vector (Z_1, Z_2) and the results in Theorems 6.1 and 6.2 also hold for the Q-class of the random vector (W_1, W_2) given $Z = z$. This follows from the observation that the joint characteristic function in the Q-class differs from the joint characteristic function $\phi(t_1, t_2)$ of (Z_1, Z_2) by the exponential of a polynomial in the variables t_1 and t_2. Similar observation holds in the conditionally independent case in Theorems 6.1 and 6.2.

(2) Rao (1971) proved that, if $X_i, 1 \leq i \leq n$ are independent random variables, then one can construct p linear forms $L_1, \ldots, L_p$ of $X_1, \ldots, X_n$ with $p(p-1)/2 \leq n \leq p(p+1)/2$ such that the joint distribution of $L_1, \ldots, L_p$ determines the distribution of $X_i, 1 \leq i \leq n$ up to Q-identical distributions. This was pointed out in Kagan and Székely (2016).

Acknowledgement : The author thanks the two reviewers for their pertinent comments and for the suggestion to introduce the concept of the Q-class for a random variable X. This work was supported under the scheme "Ramanujan Chair Professor" at the CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad 500046, India.

References:

Bairamov, I. (2011) Copula representations and order statistics for conditionally independent random variables, arXiv:11107.3200v1 [math.ST] 16 Jul 2011.

Chow, Y.S. and Teicher, H. (1978) *Probability Theory: Independence, Interchangeability, Martingales*, springer, New York.

Dawid, A.P. (1979) Conditional independence in statistical theory, *J. Roy. Statist. Soc. Ser. B*, **41**, 1-31.

Dawid, A.P. (1980) Conditional independence for statistical operations, *Ann. Statist.*, **8**, 598-617.

Kagan, A.M., Linnik, Yu.V., and Rao, C.R. (1973) *Characterization Problems in Mathematical Statistics*, Wiley, New York.

Kagan, A.M. and Székely, Gábor J. (2016) An analytic generalization of independence and identical distributiveness, *Statistics and Probability Letters*; http://dx.doi.org/10.1016/j.spl.2015.10.005.

Kotlarski, I.I. (1967) On characterizing the gamma and the normal distribution, *Pacific J. Math.*, **20**, 69-76.

Majerak, D., Nowak, W. and Zieba, W. (2005) Conditional strong law of large numbers, *Internat. J. Pure and Appl. Math.*, **20**, 143-157.

Prakasa Rao, B.L.S. (1992) *Identifiability in Stochastic Models : Characterization of Probability Distributions*, Academic Press, San Diego.

Prakasa Rao, B.L.S. (2009) Conditional independence, conditional mixing and conditional association, *Ann. Inst. Statist. Math.*, **61**, 441-460.

Prakasa Rao, B.L.S. (2013) Characterization of distributions based on functions of conditionally independent random variables, *J. Indian Statist. Assoc.*, **51**, 183-200.

Prakasa Rao, B.L.S. (2016) Characterization of probability distributions through Q-independence, *Teoriya Veryatnostei i ee Primeneniya* (to appear).

Rao, C.R. (1966) Characterization of the distribution of random variables in linear structural relations, *Sankhya*, **28**, 251-260.

Rao, C.R. (1967) On some characterization of the normal law, *Sankhya*, **29**, 1-14.

Rao, C.R. (1971) Characterization of probability laws by linear functions, *Sankhya, Series A*, **33**, 265-270.

Roussas, G.G. (2008) On conditional independence, mixing and association, *Stoch. Anal. appl.*, **26**, 1274-1309.

Sasvári, Z. (1986) Characterizing the distribution of the random variables X_1, X_2, X_3 by the distribution of $(X_1 - X_3, X_2 - X_3)$, *Prob. Theory. and Relat. Fields*, **73**, 43-49.