Author (s): B.L.S. Prakasa Rao

Title of the Report: On some maximal and integral inequalities for sub-fractional Brownian motion

Research Report No.: RR2016-05

Date: August 29, 2016
On some maximal and integral inequalities for sub-fractional Brownian motion

B.L.S. Prakasa Rao
CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad 500046, India

Abstract: We obtain a maximal inequality for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$ analogous to the Burkholder-Davis-Gundy inequality for fractional Brownian motion derived by Novikov and Valkeila (Statist. Probab. Lett. 44 (1999), 47-54) and an integral inequality for Wiener integrals with respect to a sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Keywords and phrases: Sub-fractional Brownian motion; Maximal inequality; Integral inequality; Wiener integral.

MSC 2010: 60G22.

1 Introduction

Fractional Brownian motion $W^H = \{W^H(t), t \geq 0\}$ has been used for modelling stochastic phenomena with long-range dependence. It is a centered Gaussian process with the covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

where $0 < H < 1$ and the constant $H$ is called the Hurst index. The case $H = 1/2$ corresponds to the Brownian motion. FBm is the only Gaussian process which is self-similar and has stationary increments. For properties of fBm, see Samorodnitsky and Taqqu (1994), Mishura (2008) and Prakasa Rao (2010). Bojdecki et al. (2004) introduced a centered Gaussian process $\zeta^H = \{\zeta^H(t), t \geq 0\}$ called sub-fractional Brownian motion (sub-fBm) with the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}]$$

where $0 < H < 1$. The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor (2009) introduced a Wiener integral with respect to a sub-fBm. Tudor (2007 a,b, 2008, 2009) discussed some properties
related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained. Diedhiou et al. (2011) investigated parametric estimation for stochastic differential equation (SDE) driven by a sub-fBm. Mendy (2013) studied parameter estimation for sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

\[ dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0 \]

where \( H > \frac{1}{2} \). Kuang and Xie (2013) studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Shen and Li (2014) discussed estimation for the drift of sub-fBm. Kuang and Liu (2016) discussed about the \( L^2 \)-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. (2011) obtained the Ito’s formula for sub-fractional Brownian motion with Hurst index \( H > \frac{1}{2} \).

Our interest is to obtain some maximal and integral inequalities for sub-fBm. For an overview of maximal inequalities for fBm, see Prakasa Rao (2014).

2 Preliminaries

Bojdecki et al. (2004) noted that the process

\[ \frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0, \]

where \( \{W^H(t), -\infty < t < \infty\} \) is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. They proved the following result concerning properties of a sub-fBm.

**Theorem 2.1**: Let \( \zeta^H = \{\zeta^H(t), t \geq 0\} \) be a sub-fBm. Then the following properties hold.

(i) The process \( \zeta^H \) is self-similar, that is, for every \( a > 0 \),

\[ \{a \zeta^H(at), t \geq 0\} \overset{d}= \{a^H \zeta^H(t), t \geq 0\} \]

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process \( \zeta^H \) is not Markov and it is not a semi-martingale.

(iii) For all \( s, t \geq 0 \), the covariance function \( C_H(s, t) \) of the process \( \zeta^H \) is positive for all \( s > 0, t > 0 \). Furthermore
\[ C_H(s, t) > R_H(s, t) \text{ if } H < \frac{1}{2} \]

and
\[ C_H(s, t) < R_H(s, t) \text{ if } H > \frac{1}{2}. \]

(iv) Let \( \beta_H = 2 - 2^{2H-1} \). For all \( s \geq 0, t \geq 0, \)
\[ \beta_H(t-s)^{2H} \leq E[\zeta_H^H(t) - \zeta_H^H(s)]^2 \leq (t-s)^{2H}, \text{ if } H > \frac{1}{2} \]
and
\[ (t-s)^{2H} \leq E[\zeta_H^H(t) - \zeta_H^H(s)]^2 \leq \beta_H(t-s)^{2H}, \text{ if } H < \frac{1}{2} \]
and the constants in the above inequalities are sharp.

(v) The process \( \zeta^H \) has continuous sample paths almost surely and, for each \( 0 < \epsilon < H \)
and \( T > 0 \), there exists a random variable \( K_{\epsilon,T} \) such that
\[ |\zeta_H^H(t) - \zeta_H^H(s)| \leq K_{\epsilon,T}|t-s|^{H-\epsilon}, 0 \leq s, t \leq T. \]

Let \( f : [0, T] \rightarrow \mathbb{R} \) be a measurable function and \( \alpha > 0 \), and \( \sigma \) and \( \eta \) be real. Define the Erdeyli-Kober-type fractional integral
\[
(I_{T,\sigma,\eta} f)(s) = \frac{\sigma^\alpha \eta}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-a)\eta-1} f(t)}{(t^\sigma - s^\sigma)^{1-a}} dt, s \in [0, T],
\]
and
\[
n_H(t, s) = \frac{\sqrt{\pi}}{2^{H-\frac{3}{2}}} \frac{I_{T,2^{\frac{1}{2}-H}}(u^{H-\frac{1}{2}}) I_{[0,t]}(s)}{I_H(\frac{3}{2}-H)}
\]
\[ = \frac{2^{1-H} \sqrt{\pi}}{\Gamma(H-\frac{3}{2})} \int_0^t \int_0^\infty (x^2 - s^2)^{H-\frac{3}{2}} dx I_{[0,t]}(s). \]

The following theorem is due to Dzhaparidze and Van Zanten (2004) and Tudor (2009).

**Theorem 2.2:** The following representation holds, in distribution, for the sub-fBm \( \zeta^H \):
\[
\zeta^H_t \overset{\Delta}{=} c_H \int_0^t n_H(t, s) dW_s, 0 \leq t \leq T
\]
where
\[ c_H^2 = \frac{\Gamma(2H+1) \sin(\pi H)}{\pi} \]
and \( \{W_t, t \geq 0\} \) is the standard Brownian motion.

Tudor (2007b) obtained the prediction formula for a sub-fBm. For any \( 0 < H < 1 \), and
\[
0 < a < t, \quad \text{(2.5)}
\]
\[
E[\zeta_t^H | \zeta_s^H, 0 \leq s \leq a] = S_a^H + \int_0^a \psi_{a,t}(u) d\zeta_u^H
\]
where
\[
\psi_{a,t}(u) = \frac{2 \sin(\pi(H - \frac{1}{2}))}{\pi} u(a^2 - u^2)^{-\frac{1}{2}} - \frac{1}{2} \int_a^t \frac{(z^2 - a^2)^{H - \frac{1}{2}}}{z^2 - u^2} z^{-\frac{1}{2}} dz. \quad \text{(2.6)}
\]

Let
\[
M_t^H = d_H \int_0^t s^{-H} dW_s \quad \text{where} \quad d_H = \frac{2^{H - \frac{1}{2}}}{c_H \Gamma(\frac{3}{2} - H) \sqrt{\pi}}. \quad \text{(2.7)}
\]

The process \( M_t^H = \{M_t^H, t \geq 0\} \) is a Gaussian martingale and is called the sub-fractional fundamental martingale. The filtration generated by this martingale is the same as the filtration \( \{F_t, t \geq 0\} \) generated by the sub-fBm \( \zeta_t^H \) and the quadratic variation \( \langle M_t^H, M_t^H \rangle_s \) of the martingale \( M_t^H \) over the interval \( [0, s] \) is equal to
\[
\frac{d_H^2}{2} s^{-2H} H s^{2-2H} = \lambda_H s^{2-2H} \quad \text{(say)}.
\]

For any measurable function \( f : [0, T] \rightarrow \mathbb{R} \) with \( \int_0^T f^2(s) s^{1-2H} ds < \infty \), define the probability measure \( Q_f \) by
\[
\frac{dQ_f}{dP} |_{F_t} = \exp(\int_0^t f(s) dM_s^H - \frac{1}{2} \int_0^t f^2(s) d < M_t^H > (s))
\]
\[
= \exp(\int_0^t f(s) dM_s^H - \frac{d_H^2}{2} \int_0^t f^2(s) s^{1-2H} ds).
\]

where \( P \) is the underlying probability measure. Let
\[
(\psi_H f)(s) = \frac{1}{\Gamma(\frac{3}{2} - H)} I_{0,\frac{3}{2} - H}^s f(s) \quad \text{(2.9)}
\]
where, for \( \alpha > 0 \),
\[
(I_{0,\alpha} f)(s) = \frac{\sigma s^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^s \frac{t^{\alpha-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0, T]. \quad \text{(2.10)}
\]

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor (2009)).

**Theorem 2.3:** The process
\[
\zeta_t^H - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T
\]
is a sub-fBm with respect to the probability measure $Q_f$. In particular, choosing the function $f \equiv a \in R$, it follows that the process $\{\zeta^H_t - at, 0 \leq t \leq T\}$ is a sub-fBm under the probability measure $Q_f$ with $f \equiv a \in R$.

3 Maximal inequalities

For any process $X$, defined on the underlying probability space $(\Omega, \mathcal{F}, P)$, let $X^*$ denote the supremum process defined by

$$X^*_t = \sup_{0 \leq s \leq t} |X_s|$$

whenever it is defined. Since the process $\zeta^H$ is self-similar, it follows that

$$\{\zeta^H(at), 0 \leq t \leq T\} \overset{\Delta}{=} \{a^H \zeta^H(t), 0 \leq t \leq T\}$$

for any $a > 0$ and hence

$$\zeta^{H*}(at) \overset{\Delta}{=} a^H \zeta^{H*}(t).$$

We have the following result as a consequence of the self-similarity of the process $\zeta^H$.

**Theorem 3.1:** For any $T > 0$ and $p > 0$,

$$E[(\zeta^{H*}(T))^p] = K(H, p)T^{pH}$$

where $K(H, p) = E[(\zeta^{H*}(1))^p]$.

The following theorem is due to Burkholder-Davis-Gundy (cf. Liptser and Shiryayev (1989)).

**Theorem 3.2:** Let $\{N_t, \beta_t, t \geq 0\}$ be a martingale with finite quadratic variation $\langle N, N \rangle_t$, $t \geq 0$. For any $p > 0$, and for any stopping time $\tau$, adapted to the filtration $\{\beta_t, t \geq 0\}$, there exist positive constants $c_p, C_p$ such that

$$c_p E[\langle N, N \rangle^{p/2}_\tau] \leq E[(N^*_\tau)^p] \leq C_p E[\langle N, N \rangle^{p/2}_\tau]. \tag{3. 1}$$

As an application of this result, we obtain the following inequality using the observation that the process $\{M_t, \mathcal{F}_t, t \geq 0\}$ is a martingale with quadratic variation $\langle M, M \rangle_t = \frac{d_t^2}{2-2H} t^{2-2H}$.
Theorem 3.3: For any $p > 0$ and any stopping time $\tau$ adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, there exist positive constants $c_p, C_p$ such that

\[(3.2) \quad c_p \lambda_H^{p/2} E[\tau^{p(1-H)}] \leq E[(M^*_\tau)^p] \leq C_p \lambda_H^{p/2} E[\tau^{p(1-H)}].\]

From the results in Dzhaparidze and Van Zanten (2004) and Mendy (2013), it follows that the representation

\[(3.3) \quad W_t = \int_0^t \psi_H(t, s)d\zeta^H_s\]

holds where $\{W_t, t \geq 0\}$ is a standard Brownian motion and

\[(3.4) \quad \psi_H(t, s) = \frac{s^{H-\frac{1}{2}}}{\Gamma\left(\frac{3}{2} - H\right)} [t^{H-\frac{3}{2}}(t^2 - s^2)^{-\frac{1}{2} - H} - (H - \frac{3}{2}) \int_s^t (x^2 - s^2)^{\frac{1}{2} - H} x^{-\frac{1}{2}} dx] I_{(0,t)}(s).\]

Combining the equations (2.7) and (3.3), we get that

\[(3.5) \quad M^H_t = \int_0^t k_H(t, s)d\zeta^H_s\]

where

\[(3.6) \quad k_H(t, s) = d_H s^{\frac{1}{2} - H} \psi_H(t, s)\]

and $<M,M>_t = \int_0^t \lambda_H t^{2-H} dt$. Following the technique in Novikov and Valkeila (1999), let

\[(3.7) \quad Y^H_t = \int_0^t s^{\frac{1}{2} - H} d\zeta^H_s, t \geq 0.\]

Then

\[(3.8) \quad \zeta^H_t = \int_0^t s^{H-\frac{1}{2}} dY^H_s, t \geq 0\]

and

\[(3.9) \quad M^H_t = d_H \int_0^t k_H(t, s)s^{H-\frac{1}{2}} dY_s = d_H \int_0^t \psi_H(t, s)dY_s, t \geq 0\]

Equation (3.8) implies that

\[\zeta^H_t \leq 2t^{\alpha}(Y^H_t)^\alpha\]

whenever $H > \frac{1}{2}$. Let $\alpha = H - \frac{1}{2}$. Solving the integral equation (3.9) as a generalized Abel integral equation with respect to the process $Y^H$ path-wise, we can represent the process $\{Y^H_t, t \geq 0\}$ as a stochastic integral of a function $\nu_H(t, s)$ with respect to the martingale $\{M^H_t, \mathcal{F}_t, t \geq 0\}$, that is

\[(3.10) \quad Y^H_t = \int_0^t \nu_H(t, s)dM^H_s, t \geq 0.\]
Then, it follows that
\[(Y_t^H)^* \leq \sup_{0 \leq s \leq t} |\nu_H(t, s)|(M_t^H)^*, t \geq 0.\]
Hence
\[(\zeta_t^H)^* \leq 2t^{\alpha \gamma_H^H}(M_t^H)^*, t \geq 0.\]
Let \(\gamma_t^H = \sup_{0 \leq s \leq t} |\nu_H(t, s)|.\)

Applying the inequalities given above, for any stopping time \(\tau\) with respect to the filtration \(\{\mathcal{F}_t, t \geq 0\}\), it follows that
\[(\zeta_\tau^H)^* \leq 2\tau^{\alpha \gamma_H^H}(M_\tau^H)^*.\]
Hence, for any \(p > 0\),
\[E[(\zeta_\tau^H)^*]^p \leq 2^p E[(\tau^{\alpha \gamma_H^H})^p((M_\tau^H)^*)^p] \]
Applying Holder’s inequality with \(q = \frac{H}{2\alpha} = \frac{H}{2H-1} > 1\) and \(r = \frac{H}{1-H}\), we get that
\[E[(\tau^{\alpha \gamma_H^H})^p((M_\tau^H)^*)^p] \leq (E[(\tau^{\alpha \gamma_H^H})^{pq}])^{1/q}(E[(M_\tau^H)^{pr}])^{1/r}.\]
An application of Theorem 3.2 shows that there exists a positive constant \(C_{pr}\) such that
\[E[(M_\tau^H)^{pr}] \leq C_{pr} \lambda_H^{pr/2} E[\tau^{pr(1-H)}] = C_{pr} \lambda_H^{pr/2} E[\tau^{pH}]\]
and we obtain the following theorem as a consequence of the inequalities (3.14) and (3.16).

**Theorem 3.3:** Let \(H > \frac{1}{2}\) and \(\tau\) be any stopping time adapted to filtration generated by the process \(\{\zeta_t^H, t \geq 0\}\). Then, for any \(p > 0\), there exists a positive constant \(C(p, H)\) such that
\[E[(\zeta_\tau^H)^*]^p \leq C(p, H)(E[\tau^{\alpha \gamma_H^H})^{pq}])^{1/q}(E[\tau^{pH}])^{1/r}.\]
where \(q = \frac{H}{2H-1}\) and \(r = \frac{H}{1-H}\).

A better bound can be obtained if it is possible to derive a closed form for the function \(|\nu_H(t, s)|\) and, in turn, obtain its supremum \(\gamma_t^H\) over any interval \([0, t]\).

### 4 Inequalities for Wiener integrals with respect to a sub-fBm

Tudor (2009) (cf. Mendy (2013)) has investigated properties of a Wiener integral with respect to a sub-fBm on an interval. Suppose that \(\frac{1}{2} < H < 1\). Let \(\psi\) denote the integral operator
\[\psi f(t) = H(2H-1) \int_0^T f(s)[|s-t|^{2H-2} - |s+t|^{2H-2}]ds\]
and define the inner product

\[ <f, g > = < f, \psi g > = H(2H - 1) \int_0^T \int_0^T f(s)g(t)[|s - t|^{2H-2} - |s + t|^{2H-2}]dsdt \]

where \(< . >\) denotes the usual inner product of \(L^2[0, T]\). Let \(L^2_\psi[0, T]\) be the space of equivalence classes of measurable functions such that \(< fI_{[0, T]}, fI_{[0, T]} > \psi < \infty\). The mapping \(\zeta^H_t \to I_{[0, T]}\) can be extended to an isometry between a subspace of the Gaussian space generated by the random variables \(\zeta^H_t, 0 \leq t \leq T\) and the function space \(L^2_\psi[0, T]\). For \(f \in L^2_\psi[0, T]\), define the integral

\[ \int_0^T f(s)d\zeta^H_s \]

as the image of the function \(f\) by this isometry. Note that the covariance function \(C^H(s, t)\) the sub-fBm can be represented in the form

\[ E[\zeta^H_t \zeta^H_s] = H(2H - 1) \int_0^t \int_0^s [|u - v|^{2H-2} - |u + v|^{2H-2}]dudv. \]

In general, for \(f, g \in L^2_\psi[0, T]\), it follows that

\[ E[\int_0^T f(u)d\zeta^H_u \int_0^T g(v)d\zeta^H_v] = H(2H - 1) \int_0^T \int_0^T f(u)g(v)[|u - v|^{2H-2} - |u + v|^{2H-2}]dudv \]

and

\[ E[(\int_0^T f(u)d\zeta^H_u)^2] = H(2H - 1) \int_0^T \int_0^T f(u)f(v)[|u - v|^{2H-2} - |u + v|^{2H-2}]dudv \]

We will now prove an integral inequality for a sub-fBm.

**Theorem 4.1:** Let \(\zeta^H\) be a sub-fBm with Hurst index \(H > \frac{1}{2}\). Then, for every \(r > 0\), there exists a constant \(c(H, r)\) such that,

\[ E(\int_0^T f(u)d\zeta^H_u)^r \leq c(H, r)\|f(u)\|_{L^1[0, T]}^r. \]

We will use the following result due to Hardy and Littlewood (cf. Stein (1971), Theorem 1, p.119; Mishura (2008), Theorem 1.1.1; Samko et al. (1993)) in the proof of Theorem 4.1.

**Lemma 4.2:** Let \(0 < \alpha < 1, 1 < p < \frac{1}{\alpha}\) and let \(q = \frac{p}{1-\alpha p}\). Suppose that \(f \in L^p(R)\). Then there exists a positive constant \(C_{p, q, \alpha}\) such that

\[ |\int_R (\int_R |f(u)||x - u|^{\alpha-1}du)^qdz|^{1/q} \leq C_{p, q, \alpha}(\int_R |f(u)|^p du)^{1/p}. \]
By replacing \( x \) by \(-x\) in the above inequality, it is easy to check that

(4. 7) \[
\left[ \int_R \left( \int_R |f(u)||x+u|^{\alpha-1} du \right)^q dx \right]^{1/q} \leq C_{p,q,\alpha} \left[ \int_R |f(u)|^p du \right]^{1/p}
\]

under the conditions stated in Lemma 4.2.

We will now prove Theorem 4.1.

**Proof of Theorem 4.1:** Since, the random variable \( \int_0^T f(s) d\zeta_s^H \) is a centered Gaussian random variable, for every \( r > 0 \), there exists a positive constant \( c_r \) such that

(4. 8) \[
E(\int_0^T f(u) d\zeta_u^H) = c_r [E(\int_0^T f(u) d\zeta_u^H)^2]^{1/2}.
\]

In view of the equation (4.4), the inequality (4.5) will hold if

(4. 9) \[
\int_0^T \int_0^T f(u) f(v) \left[ |u - v|^{2H-2} - |u + v|^{2H-2} \right] dudv \leq c_H \left( \int_0^T |f(u)|^{1/H} du \right)^{2H}.
\]

for some constant \( c_H > 0 \). Choose \( p = 1/H \) and \( \alpha = 2H - 1 \) in Lemma 4.2. Note that

(4. 10) \[
\int_0^T |f(u)| \left( \int_0^T |f(v)||u - v|^{2H-2} dv \right) du \leq \left( \int_0^T |f(u)|^{1/H} du \right)^H \left( \int_0^T \left( \int_0^T |f(v)||u - v|^{2H-2} dv \right)^{1-H/2} du \right) \\
\leq C_{(\frac{1}{H}, \frac{1}{H}, \alpha)} \left[ \int_0^T |f(u)|^{1/H} du \right]^{2H}.
\]

Similarly

(4. 11) \[
\int_0^T |f(u)| \left( \int_0^T |f(v)||u + v|^{2H-2} dv \right) du \leq \left( \int_0^T |f(u)|^{1/H} du \right)^H \left( \int_0^T \left( \int_0^T |f(v)||u + v|^{2H-2} dv \right)^{1-H/2} du \right) \\
\leq C_{(\frac{1}{H}, \frac{1}{H}, \alpha)} \left[ \int_0^T |f(u)|^{1/H} du \right]^{2H}.
\]

It is clear that

(4. 12) \[
\left| \int_0^T \int_0^T f(u) f(v) \left[ |u - v|^{2H-2} - |u + v|^{2H-2} \right] dv \right| du \leq \int_0^T \int_0^T |f(u)||f(v)||u - v|^{2H-2} dv \ dudv \\
+ \int_0^T \int_0^T |f(u)||f(v)||u + v|^{2H-2} dv \ dudv.
\]
Combining the above inequalities, it follows that there exists a positive constant $c_H$ such that

\[
(4.13) \quad \left| \int_0^T \int_0^T f(u)f(v)[|u-v|^{2H-2} - (u+v)^{2H-2}]dudv \right| \leq c_H \left| \int_0^T |f(u)|^{1/H}du \right|^{2H}
\]

which in turn proves the inequality (4.5).

**Acknowledgement:** This work was supported under the scheme “Ramanujan Chair Professor” at the CR Rao Advanced Institute of Mathematics, Statistics and Computer science, Hyderabad, India.

**References:**


