CRRAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)

Research Report

Author (s): **B.L.S. Prakasa Rao**

Title of the Report: **Optimal Estimation of a Signal Perturbed by a Sub-Fractional Brownian Motion**

Research Report No.: **RR2016-10**

Date: **December 7, 2016**

Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org

Optimal Estimation of a Signal Perturbed by a Sub-Fractional Brownian Motion

B.L.S. Prakasa Rao
CR Rao Advanced Institute of Mathematics, Statistics
and Computer Science, Hyderabad 500046, India

Abstract : We consider the problem of optimal estimation of the vector parameter θ of the drift term in a sub-fractional Brownian motion. We obtain the maximum likelihood estimator as well as Bayesian estimator when the prior distribution is Gaussian.

Keywords and phrases: Sub-fractional Brownian motion; Maximum likelihood estimation; Bayes estimation.

MSC 2010: 60G22.

1 Introduction

Fractional Brownian motion $W^H = \{W_t^H, t \geq 0\}$ has been used for modelling stochastic phenomena with long-range dependence. It is a centered Gaussian process with the covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

where $0 < H < 1$ and the constant H is called the Hurst index. The case $H = 1/2$ corresponds to the Brownian motion. FBm is the only Gaussian process which is self-similar and has stationary increments. For properties of fBm, see Samorodnitsky and Taqqu (1994), Mishura (2008) and Prakasa Rao (2010). Bojdecki et al. (2004) introduced a centered Gaussian process $\zeta^H = \{\zeta_t^H, t \geq 0\}$ called *sub-fractional Brownian motion* (sub-fBm) with the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}]$$

where $0 < H < 1$. The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor (2009) introduced a Wiener integral with respect to a sub-fBm. Tudor (2007 a,b, 2008, 2009) discussed some properties related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained. Diedhiou et al. (2011) investigated parametric estimation for a stochastic differential equation (SDE) driven by a sub-fBm. Mendy (2013) studied parameter estimation for the sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dX_t = \theta X_t dt + d\zeta_t^H, t \geq 0$$

where $H > \frac{1}{2}$. Kuang and Xie (2013) studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Shen and Yan (2014) discussed estimation for the drift of a sub-fBm. Kuang and Liu (2016) discussed about the L^2-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. (2011) obtained the Ito's formula for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$. For results on some maximal and integral inequalities for sub-fractional Brownian motion, see Prakasa Rao (2016).

For a discussion on methods of statistical inference for estimation of parameters for processes driven by a fractional Brownian motion, see Prakasa Rao (2010).

2 Preliminaries

Bojdecki et al. (2004) noted that the process

$$\frac{1}{\sqrt{2}}[W_t^H + W_{-t}^H], t \geq 0,$$

where $\{W_t^H, -\infty < t < \infty\}$ is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. They proved the following result concerning properties of a sub-fBm.

Theorem 2.1: Let $\zeta^H = \{\zeta_t^H, t \geq 0\}$ be a sub-fBm defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$.. Then the following properties hold.

(i) The process ζ^H is self-similar, that is, for every $a > 0$,

$$\{\zeta_{at}^H, t \geq 0\} \triangleq \{a^H \zeta_t^H, t \geq 0\}$$

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process ζ^H is not Markov and it is not a semi-martingale.

(iii) For all $s, t \geq 0$, the covariance function $C_H(s,t)$ of the process ζ^H is positive for all $s > 0, t > 0$. Furthermore

$$C_H(s,t) > R_H(s,t) \text{ if } H < \frac{1}{2}$$

and

$$C_H(s,t) < R_H(s,t) \text{ if } H > \frac{1}{2}.$$

(iv) Let $\beta_H = 2 - 2^{2H-1}$. For all $s \geq 0, t \geq 0$,

$$\beta_H(t-s)^{2H} \leq E[\zeta_t^H - \zeta_s^H]^2 \leq (t-s)^{2H}, \text{ if } H > \frac{1}{2}$$

and

$$(t-s)^{2H} \leq E[\zeta_t^H - \zeta_s^H]^2 \leq \beta_H(t-s)^{2H}, \text{ if } H < \frac{1}{2}$$

and the constants in the above inequalities are sharp.

(v) The process ζ^H has continuous sample paths almost surely and, for each $0 < \epsilon < H$ and $T > 0$, there exists a random variable $K_{\epsilon,T}$ such that

$$|\zeta_t^H - \zeta_s^H| \leq K_{\epsilon,T}|t-s|^{H-\epsilon}, 0 \leq s,t \leq T \text{ a.s.}$$

Let $f : [0,T] \to R$ be a measurable function and $\alpha > 0$, and σ and η be real. Define the Erdeyli-Kober-type fractional integral

$$(2.\ 1) \qquad (I^{\alpha}_{T,\sigma,\eta} f)(s) = \frac{\sigma s^{\alpha\eta}}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0,T],$$

and

$$(2.\ 2) \qquad \begin{aligned} n_H(t,s) &= \frac{\sqrt{\pi}}{2^{H-\frac{1}{2}}} I^{H-\frac{1}{2}}_{T,2,\frac{3-2H}{4}}(u^{H-\frac{1}{2}}) I_{[0,t)}(s) \\ &= \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{\frac{3}{2}-H} \int_0^t (x^2 - s^2)^{H-\frac{3}{2}} dx \ I_{(0,t)}(s). \end{aligned}$$

The following theorem is due to Dzhaparidze and Van Zanten (2004) and Tudor (2009).

Theorem 2.2: The following representation holds, in distribution, for the sub-fBm ζ^H:

$$(2.\ 3) \qquad \zeta_t^H \triangleq c_H \int_0^t n_H(t,s) dW_s, 0 \leq t \leq T$$

where

$$(2.\ 4) \qquad c_H^2 = \frac{\Gamma(2H+1)\ \sin(\pi H)}{\pi}$$

and $\{W_t, t \geq 0\}$ is the standard Brownian motion.

Tudor (2007b) obtained the prediction formula for a sub-fBm. For any $0 < H < 1$, and $0 < a < t$,

$$E[\zeta_t^H | \zeta_s^H, 0 \leq s \leq a] = \zeta_a^H + \int_0^a \psi_{a,t}(u) d\zeta_u^H \tag{2. 5}$$

where

$$\psi_{a,t}(u) = \frac{2\ \sin(\pi(H - \frac{1}{2}))}{\pi} u(a^2 - u^2)^{\frac{1}{2} - H} \int_a^t \frac{(z^2 - a^2)^{H - \frac{1}{2}}}{z^2 - u^2} z^{H - \frac{1}{2}} dz. \tag{2. 6}$$

Let

$$M_t^H = d_H \int_0^t s^{\frac{1}{2} - H} dW_s \tag{2. 7}$$

where

$$d_H = \frac{2^{H - \frac{1}{2}}}{c_H \Gamma(\frac{3}{2} - H)\sqrt{\pi}}. \tag{2. 8}$$

The process $M^H = \{M_t^H, t \geq 0\}$ is a Gaussian martingale (cf. Tudor (2009)) and is called the *sub-fractional fundamental martingale*. The filtration generated by this martingale is the same as the filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the sub-fBm ζ^H and the quadratic variation $< M^H, M^H >_s$ of the martingale M^H over the interval $[0, s]$ is equal to $w^H(s) = \frac{d_H^2}{2-2H} s^{2-2H} = \lambda_H s^{2-2H}$ (say). For any measurable function $f : [0, T] \to R$ with $\int_0^T f^2(s) s^{1-2H} ds < \infty$, define the probability measure Q_f by

$$\begin{aligned}
\frac{dQ_f}{dP}|_{\mathcal{F}_t} &= \exp(\int_0^t f(s) dM_s^H - \frac{1}{2} \int_0^t f^2(s) d < M^H, M^H > (s)) \\
&= \exp(\int_0^t f(s) dM_s^H - \frac{d_H^2}{2} \int_0^t f^2(s) s^{1-2H} ds).
\end{aligned}$$

where P is the underlying probability measure. Let

$$(\psi_H f)(s) = \frac{1}{\Gamma(\frac{3}{2} - H)} I_{0,2,\frac{1}{2} - H}^{H - \frac{1}{2}} f(s) \tag{2. 9}$$

where, for $\alpha > 0$,

$$(I_{0,\sigma,\eta}^{\alpha} f)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^s \frac{t^{\sigma(1+\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0, T]. \tag{2. 10}$$

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor (2009)).

Theorem 2.3: The process

$$\zeta_t^H - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T$$

is a sub-fbm with respect to the probability measure Q_f. In particular, choosing the function $f \equiv a \in R$, it follows that the process $\{\zeta_t^H - at, 0 \leq t \leq T\}$ is a sub-fBm under the probability measure Q_f with $f \equiv a \in R$.

Let

$$\psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)}[t^{H-\frac{1}{2}}(t^2-s^2)^{\frac{1}{2}-H} - (H-\frac{3}{2})\int_s^t (x^2-s^2)^{\frac{1}{2}-H}x^{H-\frac{3}{2}}dx]I_{(0,t)}(s)$$

and

(2. 11) $$k_H(t,s) = d_H s^{\frac{1}{2}-H}\psi_H(t,s).$$

Let $Y = \{Y_t, t \geq 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ and suppose the process Y satisfies the stochastic differential equation

(2. 12) $$dY_t = C(t)dt + D(t)d\zeta_t^H, t \geq 0$$

where the function $C(t)$, adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, and the non-random function $D(t)$ are such that the process

(2. 13) $$R_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{D(s)}ds, t \geq 0$$

is well-defined and the derivative is understood in the sense of absolute continuity with respect to the measure generated by the function w_H. Differentiation with respect to w_t^H is understood in the sense:

$$dw_t^H = \lambda_H(2-2H)t^{1-2H}dt$$

and

$$\frac{df(t)}{dw_t^H} = \frac{df(t)}{dt}/\frac{dw_t^H}{dt}.$$

Suppose the process $R_H(t)$ defined over the interval $[0,T]$ belongs to the space $L^2([0,T], dw_t^H)$. Define

(2. 14) $$\Lambda_t^H = \exp\{\int_0^t R_H(s)dM_s^H - \frac{1}{2}\int_0^t [R_H(s)]^2 dw_s^H\}.$$

If $E(\Lambda_T^H) = 1$, then the measure $P^Y = \Lambda_T^H.P$ is a probability measure and the probability distribution of the process Y under P^Y coincides with the distribution of the process $\int_0^\cdot D(s)d\zeta_s^H$ with respect to P (cf. Tudor (2009)).

We call the process Λ^H as the *likelihood process* or the Radon-Nikodym derivative $\frac{dP^Y}{dP}$ of the measure P^Y with respect to the measure P(cf. Tudor (2009)).

Let $\xi = \{\xi_t, t \geq 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ defined by the equation

$$\xi_t = a(t) + \eta(t)\zeta_t^H, t \geq 0 \tag{2. 15}$$

where ζ^H is a sub-fractional Brownian motion as defined above with Hurst index $H \in (\frac{1}{2}, 1)$, the drift coefficient is given by

$$a(t) = \sum_{i=1}^{k} \theta_i \tau_i(t) \tag{2. 16}$$

where the vector parameter $\theta = (\theta_1, \ldots, \theta_k)$ is *unknown* but the function $\tau(t) = (\tau_1(t), \ldots, \tau_k(t))$ is known and the function $\eta(t)$ is assumed to be *known*. The problem of estimation of the parameter θ, given the observations $\{\xi_s, 0 \leq s \leq t\}$, has been investigated by Diedhiou et al. (2011) when $k = 1$. We consider the problem of sequential estimation of the vector parameter θ given the observations $\{\xi_s, 0 \leq s \leq t\}$ available up to time t using the maximum likelihood and Bayesian methods. Sequential estimation and testing for parameters, for processes driven by a fractional Brownian motion, were investigated in Prakasa Rao (2003, 2004, 2005). For a survey of problems of estimation for fractional diffusion processes, see Prakasa Rao (2010) and for diffusion processes, see Prakasa Rao (1999). Optimal estimation of a signal perturbed by a fractional Brownian noise has been recently discussed by Artemov and Burnaev (2016). Bayesian sequential estimation of the drift parameter of fractional Brownian motion is also investigated in Cetin et al. (2013). We now consider a stochastic differential equation of the form

$$d\xi_t = \sum_{i=1}^{k} \theta_i \phi_i(t)dt + \sigma(t)d\zeta_t^H, t \geq 0 \tag{2. 17}$$

and discuss the problem of estimation of the vector parameter $\theta = (\theta_1, \ldots, \theta_k)$ based on the observation of the process ξ over the interval $[0, t]$.

3 Maximum likelihood estimation of the drift parameter

We will now investigate the maximum likelihood estimation of the parameter $\theta = (\theta_1, \ldots, \theta_k)$ based on the observation of the process $\{\xi_t, 0 \leq t \leq T\}$. Since the filtrations generated by the processes $\{\xi_t, 0 \leq t \leq T\}$, $\{\zeta_t^H, 0 \leq t \leq T\}$ and $\{M_t^H, 0 \leq t \leq T\}$ are the same, the information contained in the three sets of observations is the same and hence the problem of estimation of the parameter θ based on the observations $\{\xi_t, 0 \leq t \leq T\}$ is equivalent to the problem of estimation based on the the process $\{M_t^H, 0 \leq t \leq T\}$. Following the general form of the process $R_H(t)$ defined in the previous section, we define

$$R_H(t) = \sum_{i=1}^{k} \theta_i \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{\phi_i(s)}{\sigma(s)} ds = \sum_{i=1}^{k} \theta_i \psi_i(t) \tag{3. 1}$$

where

$$\psi_i(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{\phi_i(s)}{\sigma(s)} ds, 1 \le i \le k. \tag{3. 2}$$

and the function $k_H(t,s)$ is as defined by (2.12). The likelihood process Λ^H is given by the equation

$$\Lambda_t^H(\theta) = \exp\{\sum_{i=1}^{k} \theta_i \int_0^t \psi_i(s) dM_s^H - \frac{1}{2} \int_0^t [\sum_{i=1}^{k} \theta_i \psi_i(s)]^2 dw_s^H\}. \tag{3. 3}$$

Let $J_H(t)$ denote the matrix of order $k \times k$ with the (i,j)-th element

$$(J_H(t))_{(i,j)} = \int_0^t \psi_i(s) \psi_j(s) dw_s^H \tag{3. 4}$$

and let $\psi^H = \{\psi_t^H, t \ge 0\}$ be a k-dimensional process with the i-th component of ψ_t^H as

$$(\psi_t^H)_i = \int_0^t \psi_i(s) dM_s^H, 1 \le i \le k. \tag{3. 5}$$

Following the notation defined above, the likelihood process can be written in the form

$$\Lambda_t^H(\theta) = \exp\{\theta' \psi_t^H - \frac{1}{2} \theta' J_H(t) \theta\}. \tag{3. 6}$$

The maximum likelihood estimator $\hat{\theta}_t$ of the parameter θ is a maximizer of the likelihood $\Lambda_s^H(\theta)$ over the interval $[0,t]$ and can be obtained as a solution of the system of linear equations

$$\int_0^t \psi_i(s) dM_s^H - \sum_{j=1}^{k} \theta_j \int_0^t \psi_i(s) \psi_j(s) dw_s^H = 0, 1 \le i \le k \tag{3. 7}$$

which, in turn, can be written in the form

$$\psi_t^H - J_H(t) \theta = 0. \tag{3. 8}$$

If the matrix $J_H(t)$ is invertible, then the maximum likelihood estimator (MLE) of the vector parameter θ is given by the equation

$$\hat{\theta}_t = J_H^{-1}(t) \psi_t^H. \tag{3. 9}$$

Let θ^0 be the true mean vector. Note that the martingale M^H is a zero mean Gaussian martingale and hence the random vector ψ_t^H has the multivariate normal distribution. This in turn will imply that the the random vector $(\hat{\theta}_t - \theta_0)$ has the multivariate normal distribution with mean zero and the covariance matrix $J_H^{-1}(t)$.

4 Bayes estimation of the drift parameter

We now consider the problem of Bayes estimation of the parameter $\theta \in R^k$ assuming that the parameter θ has a prior probability measure with density $p^\theta(.)$ with respect to the Lebesgue measure on R^k and the loss function is the squared error loss function. It is well known that the Bayes estimator is the conditional expectation of the parameter given the observed data, that is, it is the mean or expectation of the posterior distribution of the parameter θ given the observed data. The posterior density of θ given the observed data $\{\xi_s, 0 \leq s \leq t\}$ or equivalently the information $\mathcal{F}_t$, the σ-algebra generated by the family $\{\xi_s, 0 \leq s \leq t\}$, is given by

(4. 1)
$$p^\theta(z|\mathcal{F}_t) = \frac{p^\theta(z)\Lambda_t^H(z)}{\int_{R^k} p^\theta(y)\Lambda_t^H(y)dy}, z \in R^k$$

where $\Lambda_t^H(z)$ is the likelihood process defined earlier. We will also consider the problem of finding the optimal sequential Bayes estimation rule $\tilde{\delta} = (\tilde{\tau}, \tilde{\theta}_\tau)$ for estimation of the parameter θ in the sense that

(4. 2)
$$\inf_{\delta \in \mathcal{D}} E[c\tau + ||\theta_\tau^* - \theta||^2] = E[c\tilde{\tau} + ||\theta - \tilde{\theta}_{\tilde{\tau}}||^2]$$

where $\mathcal{D} = \{\delta : (\tau, \theta_\tau^*)\}$ is a class of stopping rules with finite stopping time $\tau \leq T < \infty$ with respect to the filtration $\{\mathcal{F}_t, 0 \leq s \leq t\}$and estimate the parameter θ by θ_τ^*. Here $\mathcal{F}_t$ is the σ-algebra generated by the process $\{\xi_s, 0 \leq s \leq t\}$. The constant $c > 0$ can be interpreted as the cost per unit of observation and the Bayes sequential estimation consists in stopping sampling at time $\tilde{\tau}$ and declaring $\tilde{\theta}_{\tilde{\tau}}$ as the optimal estimator of θ.

Special case: Suppose the vector θ has a multivariate normal prior density with the mean vector $\mathbf{m}$ and the covariance matrix Σ. Following the standard methods, it can be shown that the optimal Bayes estimator $\tilde{\theta}_t$, under the squared error loss function based on the observations up to time t, is given by

(4. 3)
$$\tilde{\theta}_t = E[\theta|\mathcal{F}_t] = (J_H(t) + \Sigma^{-1})^{-1}(\psi_t^H + \Sigma^{-1}\mathbf{m})$$

and the mean squared error $E[||\theta - \tilde{\theta}_t||^2|\mathcal{F}_t]$ is the trace of the posterior covariance matrix given by

(4. 4)
$$Cov[\theta|\mathcal{F}_t] = (J_H(t) + \Sigma^{-1})^{-1}.$$

This can be checked by the arguments similar to those given in the proof of Theorem 3 in Artemov and Burnaev (2016). We omit the details. The optimal stopping time in this special case is given by

$$\tilde{\tau} = \arg\inf_{\tau\in\mathcal{D}} E[c\tau + E(||\theta-\tilde{\theta}_\tau||^2|\mathcal{F}_\tau)] = \arg\inf_{t\in[0,T]} F_H(t) \tag{4. 5}$$

where

$$F_H(t) = ct + E(||\theta-\tilde{\theta}||^2|\mathcal{F}_t) = ct + tr((J_H(t)+\Sigma^{-1})^{-1}), 0\leq t\leq T. \tag{4. 6}$$

Note that the function $F_H(t)$ is deterministic and hence the optimal stopping rule is deterministic in this special case.

Suppose the observation process $\xi = \{\xi_t, t\geq 0\}$ satisfies the stochastic differential equation

$$d\xi_t = \theta dt + \sigma d\zeta_t^H, t\geq 0 \tag{4. 7}$$

where θ is a scalar and is normally distributed a priori with mean m and variance γ^2, then the posterior distribution of θ given the observed data $\{\xi_s, 0\leq s\leq t\}$ is normal with the mean

$$\frac{(M_t^H/\sigma)+(m/\gamma^2)}{w_H(t)/\sigma^2+1/\gamma^2}$$

and the variance

$$\frac{1}{(w_H(t)/\sigma^2)+(1/\gamma^2)}.$$

From the general results on Bayes estimation for squared error loss function, it follows that the Bayes estimator for the parameter θ is given by

$$\tilde{\theta} = E[\theta|\mathcal{F}_t] = \frac{(M_t^H/\sigma)+(m/\gamma^2)}{w_H(t)/\sigma^2+1/\gamma^2} \tag{4. 8}$$

and the variance of this estimator is

$$E[(\theta-\tilde{\theta})^2|\mathcal{F}_t] = \frac{1}{(w_H(t)/\sigma^2)+(1/\gamma^2)}. \tag{4. 9}$$

Remarks: It is possible to investigate the problem of Bayes estimation for the vector parameter $\theta\in R^k$ when it has a uniform prior on the k-dimensional cube $\Pi_{i=1}^k[a_i,b_i]$ following the arguments in Artemov and Burnaev (2016). We will not go into the details here.

Acknowledgement: This work was supported under the scheme "Ramanujan Chair Professor" at the CR Rao Advanced Institute of Mathematics, Statistics and Computer science, Hyderabad, India.

References:

Artemov, A.V. and Burnaev, E.V. (2016) Optimal estimation of a signal perturbed by a fractional Brownian noise, *Theory Probab. Appl.*, **60**, 126-134.

Bojdecki, T. and Gorostiza, A. Talarczyk (2004) Sub-fractional Brownian motion and its relation to occupation times, *Statist. Probab. Lett.*, **69**, 405-419.

Cetin, U., Novikov, A. and Shiryayev, A.N. (2013) Bayesian sequential estimation of a drift of fractional Brownian motion, *Sequential Anal.*, **32**, 288-296.

Diedhiou, A., Manga, C. and Mendy, I. (2011) Parametric estimation for SDEs with additive sub-fractional Brownian motion, *Journal of Numerical Mathematics and Stochastics*, **3**, 37-45.

Dzhaparidze, K. and Van Zanten, H. (2004) A series expansion of fractional Brownian motion, *Probab. Theory Related Fields*, **103**, 39-55.

Kuang, Nenghui and Liu, Bingquan (2016) Parameter estimations for the sub-fractional Brownian motion with drift at discrete observation, *Brazilian Journal of Probability and Statistics* (to appear).

Kuang, N.H. and Xie, H.T. (2013) Maximum likelihood estimator for the sub-fractional Brownian motion approximated by a random walk, *Ann. Inst. Statist. Math.*, DOI: 10.1007/s10463-013-0439-4.

Liptser, R.S. and Shiryayev, A.N. (1989) *The Theory of Martingales*, Kluwer, Dordrecht.

Mendy, I. (2013) Parametric estimation for sub-fractional Ornstein-Uhlenbeck process, *J. Stat. Plan. Infer.*, **143**, 663-674.

Mishura, Yulia (2008) *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Lecture Notes in Mathematics No. 1929, Springer-Verlag, Berlin.

Novikov, A. and Valkeila, E. (1999) On some maximal inequalities for Fractional Brownian motion, *Statist. Probab. Lett.*, **44**, 47-54.

Prakasa Rao, B.L.S. (1999) *Statistical Inference for Diffusion Processes*, Arnold, London

Prakasa Rao, B.L.S. (2003) Parameter estimation for linear stochastic differential equations driven by fractional Brownian motion, *Random oper. Stochastic equations*, **11**, 229-242.

Prakasa Rao, B.L.S. (2004) Sequential estimation for fractional Ornstein-Uhlenbeck type process, *Sequential Anal.*, **23**, 33-44.

Prakasa Rao, B.L.S. (2005) Sequential testing for simple hypotheses for processes driven by fractional Brownian motion, *Sequential Anal.*, **24**, 189-203.

Prakasa Rao, B.L.S. (2005) Estimation for translation of a process driven by fractional Brownian motion, *Stochastic Anal. Appl.*, **23**, 1199-1212.

Prakasa Rao, B.L.S. (2010) *Statistical Inference for Fractional Diffusion Processes*, Wiley, London.

Prakasa Rao, B.l.S. (2014) Maximal inequalities for fractional Brownian motion: an overview, *Stochastic Anal. and Appl.*, **32**, 450-479.

Prakasa Rao, B.L.S. (2016) On some maximal and integral inequalities for sub-fractional Brownian motion, *Stochastic Anal. Appl.* (to appear).

Samko, S.G., Kilbas, A.A., and Marichev, O.I. (1993) *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Science Publishers, New York.

Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes*, Chapman and Hall, New York.

Shen, G.J. and Yan, L.T. (2014) Estimators for the drift of subfractional Brownian motion, *Communications in Statistics-Theory and Methods*, **43**, 1601-1612.

Stein, E.M. (1971) *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ.

Tudor, Constantin (2007a) Some properties of the sub-fractional Brownian motion, *Stochastics*, **79**, 431-448.

Tudor, Constantin (2007b) Prediction and linear filtering with sub-fractional Brownian motion, Preprint.

Tudor, Constantin (2008) Some aspects of stochastic calculus for the sub-fractional Brownian motion, *Analele Universitat ii Bucaresti, Matematica*, Anul LVII, pp. 199-230.

Tudor, Constantin (2009) On the Wiener integral with respect to a sub-fractional Brownian motion on an interval, *J. Math. Anal. Appl.*, **351**, 456-468.

Yan, Litan., Guangjun and Kun He (2011) Ito's formula for a sub-fractional Brownian motion, *Communications of Stochastic Analysis*, **5**, 135-159.