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FOR LINEAR FORMS OF Q-INDEPENDENT  
RANDOM SEQUENCES**

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**ON THE SKITOVICH-DARMOIS-RAMACHANDRAN-IBRAGIMOV  
THEOREM FOR LINEAR FORMS OF  $Q$ -INDEPENDENT  
RANDOM SEQUENCES**

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**Abstract:** We characterize the normal distribution based on the  $Q$ -independence of linear forms based on infinite sequences of  $Q$ -independent random variables.

**Mathematics Subject Classification:** 60E10.

**Keywords:**  $Q$ -independence;  $Q$ -identically distributiveness; Linear forms; Normal distribution; Skitovich-Darmois-Ramachandran-Ibragimov theorem.

## 1 Introduction

If  $\psi$  and  $\eta$  are independent random variables, then the independence of the random variables  $\psi + \eta$  and  $\psi - \eta$  implies that  $\psi$  and  $\eta$  are normally distributed (cf. Bernstein (1941), Cramer (1936)). This result was generalized to linear forms of independent random variables by Skitovich (1953, 1954) and Darmois (1953). Let  $\psi_1, \dots, \psi_n$  be independent random variables and let  $a_i, 1 \leq i \leq n; b_i, 1 \leq i \leq n$  be nonzero real numbers. Define

$$L_1 = \sum_{j=1}^n a_j \psi_j \text{ and } L_2 = \sum_{j=1}^n b_j \psi_j.$$

Skitovich and Darmois proved independently that, if  $L_1$  and  $L_2$  are independent, then the random variables  $\psi_j, 1 \leq j \leq n$  are normally distributed. Ramachandran (1967) extended Skitovich and Darmois's result to the case where the number of summands is infinite and both the sequences  $\{a_j b_j^{-1}, j \geq 1\}$  and  $\{a_j^{-1} b_j, j \geq 1\}$  are bounded. Ibragimov (2013) proved that the result will hold if at least one of the sequences  $\{a_j b_j^{-1}, j \geq 1\}, \{a_j^{-1} b_j, j \geq 1\}$  is bounded.

Kagan and Székely (2016) introduced the notion of  $Q$ -independence and  $Q$ -identically distributed random variables and proved that the classical characterization properties of normal distribution due to Cramér (1936), Darmois-Skitovich (Darmois (1953); Skitovich

(1953, 1954)), Marcinkiewicz (1938) and Vershik (1964) continue to hold for  $Q$ -independent random variables. We will now characterize the family of distributions based on linear forms of infinite sequences of random variables with  $Q$ -independence property to be defined in the next section extending the results of Ibragimov (2013) and Ramachandran (1967).

Characterizations of probability distributions through linear forms based on finite sequences of random variables with  $Q$ -independence or conditional  $Q$ -independence are discussed in Prakasa Rao (2016a,b) extending the results in Kotlarski (1966, 1967), Prakasa Rao (1968), Miller (1970), Rao (1971) and Sasvári (1986) (cf. Prakasa Rao (1992)) for independent random variables.

## 2 $Q$ -independence

Let  $X_1, \dots, X_n$  be random variables and let the characteristic function of  $X_i$  be  $\phi_i(t)$  for  $i = 1, \dots, n$ . Following Kagan and Székely (2016), the collection  $X_i, 1 \leq i \leq n$ , is said to be  $Q$ -independent if the joint characteristic function of  $(X_1, \dots, X_n)$  can be represented as

$$\begin{aligned} \phi_{X_1, \dots, X_n}(t_1, \dots, t_n) &\equiv E[\exp(it_1 X_1 + \dots + it_n X_n)] \\ &= \prod_{j=1}^n \phi_j(t_j) \exp\{q(t_1, \dots, t_n)\}, t_1, \dots, t_n \in R \end{aligned}$$

where  $q(t_1, \dots, t_n)$  is a polynomial in  $t_1, \dots, t_n$ .

The random variables  $X_j$  and  $X_k$  are said to be  $Q$ -identically distributed if

$$\phi_j(t) = \phi_k(t) \exp\{q(t)\}$$

where  $q(\cdot)$  is a polynomial.

An infinite sequence of random variables  $\{X_i, i \geq 1\}$  is said to be  $Q$ -independent if all finite subsets of the sequence  $\{X_i, i \geq 1\}$  are  $Q$ -independent.

It is known that two random variables could be  $Q$ -independent but not independent. For instance if  $X, Y, Z$  are non-degenerate independent Gaussian random variables, then  $X + Y$  and  $X + Z$  are  $Q$ -independent but not independent (Kagan and Székely (2016)).

Kagan and Székely (2016) obtained the following theorem characterizing the normal distribution through linear forms of  $Q$ -independent random variables generalizing the results due to Skitovich (1953,1954), Darmois (1953), Marcinkiewicz (1938) and Vershik (1964).

**Theorem 2.1:** (i) Let  $X_1, \dots, X_n$  be  $Q$ -independent random variables and define  $L = \sum_{j=1}^n a_j X_j$  with nonzero  $a_j, 1 \leq j \leq n$ . If the random variable  $L$  has a normal distribution, then each of the random variables  $X_j, 1 \leq j \leq n$  has the normal distribution.

(ii) Let  $X_1, \dots, X_n$  be  $Q$ -independent random variables and define  $L_1 = \sum_{j=1}^n a_j X_j$  and  $L_2 = \sum_{j=1}^n b_j X_j$ . Suppose that the random variables  $L_1$  and  $L_2$  are  $Q$ -independent. Then each of  $X_j$ , with  $a_j b_j \neq 0$ , will have the normal distribution.

(iii) Let  $X_1, \dots, X_n$  be  $Q$ -independent random variables and define  $L_1 = \sum_{j=1}^n a_j X_j$  and  $L_2 = \sum_{j=1}^n b_j X_j$ . Suppose that  $E|X_i|^m < \infty, m \geq 1$  and the random variables  $L_1$  and  $L_2$  are  $Q$ -identically distributed satisfying the conditions

$$|a_i| > \max\{|a_1|, \dots, |a_{i-1}|, |a_{i+1}|, \dots, |a_n|, |b_1|, \dots, |b_n|\}$$

for some  $i$  with  $1 \leq i \leq n$ , then every  $X_j, 1 \leq j \leq n$  has the normal distribution.

(iv) Let  $\mathbf{X}$  be a random vector with finite covariance matrix  $V$  with  $\text{rank } V \geq 2$ . If uncorrelatedness of two linear forms  $L_1 = \mathbf{a}'\mathbf{X}$  and  $L_2 = \mathbf{b}'\mathbf{X}$  implies their  $Q$ -independence, then the random vector  $\mathbf{X}$  is multivariate normal.

### 3 Main result

Our aim is to obtain a characterization of the normal distribution through  $Q$ -independence of linear forms of  $Q$ -independent random variables when the number of summands is infinite generalizing the work of Ramachandran (1967) and Ibragimov (2013).

Let  $\{X_i, i \geq 1\}$  be an infinite sequence of  $Q$ -independent random variables. Consider the linear forms

$$L_1 = \sum_{j=1}^{\infty} a_j X_j \quad \text{and} \quad L_2 = \sum_{j=1}^{\infty} b_j X_j$$

where  $a_j, b_j, j \geq 1$  are non-zero real numbers. Suppose that the series  $L_1$  and  $L_2$  converge almost surely, the random variables  $L_1$  and  $L_2$  are  $Q$ -independent and at least one of the sequences  $\{a_j b_j^{-1}, j \geq 1\}, \{a_j^{-1} b_j, j \geq 1\}$  is bounded.

Since the series  $L_1$  and  $L_2$  are assumed to converge almost surely, it follows that their distributions of the partial sums of the series  $L_1$  and  $L_2$  converge weakly to the distributions

of  $L_1$  and  $L_2$  respectively which in turn implies that the characteristic functions of the partial sums converge to the corresponding limiting characteristic functions and this convergence is uniform on any finite interval. Since the characteristic function of the random variable  $L_1$  is non-zero in an interval around the origin, it follows that there exists an interval around the origin in which none of the characteristic functions of  $X_j$  vanish

Without loss of generality, suppose that

$$L_1 = \sum_{j=1}^{\infty} X_j, L_2 = \sum_{j=1}^{\infty} c_j X_j$$

and that the sequence  $\{c_j\}$  is bounded.

**Theorem 3.1:** Suppose the random variables  $L_1$  and  $L_2$  are  $Q$ -independent and let  $f_j(t)$  denote the characteristic function of the random variable  $X_j$  for  $j \geq 1$ . Suppose characteristic functions of the random variables  $L_i, i = 1, 2$  satisfy the property

$$E[\exp(itL_1)] = \prod_{j=1}^{\infty} f_j(t) \exp[q_1(t)]$$

and

$$E[\exp(itL_2)] = \prod_{j=1}^{\infty} f_j(c_j t) \exp[q_2(t)]$$

where  $q_i(t), i = 1, 2$  are polynomials in  $t$ . Then the random variables  $\{X_j, j \geq 1\}$  are normally distributed.

**Proof:** Let  $\phi_i(t)$  denote the characteristic function of  $L_i$  for  $i = 1, 2$ . Arguments presented earlier imply that there exists an interval  $I$  around the origin such that  $f_j(t) \neq 0$  and  $f_j(c_j t) \neq 0$  for all  $t \in I$  and for all  $j \geq 1$ . Since  $L_1$  and  $L_2$  are  $Q$ -independent, it follows that

$$E[e^{itL_1 + isL_2}] = E[e^{itL_1}]E[e^{isL_2}] \exp[q_0(t, s)]$$

where  $q_0(t, s)$  is a polynomial in  $t, s$ . Note that

$$E[e^{itL_1}] = E[e^{it \sum_{j=1}^{\infty} X_j}] = \prod_{j=1}^{\infty} f_j(t) \exp[q_1(t)]$$

where  $q_1(t)$  is a polynomial in  $t$ . Similarly

$$\begin{aligned} E[e^{itL_2}] &= E[e^{it \sum_{j=1}^{\infty} c_j X_j}] \\ &= \prod_{j=1}^{\infty} f_j(c_j t) \exp[q_2(t)] \end{aligned}$$

and

$$\begin{aligned} E[e^{itL_1+isL_2}] &= E[e^{i\sum_{j=1}^{\infty}(t+c_j s)X_j}] \\ &= \prod_{j=1}^{\infty} f_j(t+c_j s) \exp[q_3(t, s)] \end{aligned}$$

where  $q_3(t, s)$  is a polynomial in  $t$  and  $s$ . Combining the above relations, it follows that

$$\begin{aligned} &\prod_{j=1}^{\infty} f_j(t+c_j s) \exp[q_3(t, s)] \\ &= \prod_{j=1}^{\infty} f_j(t) \exp[q_1(t)] \prod_{j=1}^{\infty} f_j(c_j s) \exp[q_2(s)] \exp[q_0(t, s)]. \end{aligned}$$

which implies that

$$(3.1) \quad \prod_{j=1}^{\infty} f_j(t) \prod_{j=1}^{\infty} f_j(c_j s) = \prod_{j=1}^{\infty} f_j(t+c_j s) \exp[q(t, s)]$$

where  $q(t, s)$  is a polynomial in  $t, s$ . The infinite products

$$\prod_{j=1}^{\infty} f_j(t), \prod_{j=1}^{\infty} f_j(c_j s) \text{ and } \prod_{j=1}^{\infty} f_j(t+c_j s)$$

exist and are nonzero as the arguments given in Lemma 3.2 prove. We now adapt the methods in Ibragimov (2013) and Ramachandran (1967). It can be seen that if  $f_j(t), j \geq 1$  satisfy the above equation, the symmetrical characteristic functions  $g_j(t) = f_j(t)f_j(-t) = |f_j(t)|^2, j \geq 1$  will also satisfy similar equation. If we can show that all the functions  $g_j(t)$  are the characteristic functions of the normal distribution, then, from the Cramer's theorem (Cramer (1936)), it follows that all the function  $f_j(t)$  are the characteristic functions of the normal distribution. Here after we will assume that the random variables  $X_j$  are symmetrically distributed and hence the functions  $f_j$  are non-negative and  $f_j(t) = f_j(-t), t \geq 0, j \geq 1$ .

**Lemma 3.2:** All solutions  $f_j(t)$  of the equation (3.1) are strictly positive, that is,  $f_j(t) > 0$  for all  $t \in R$ .

**Proof:** Note that the function  $\phi_1(t) = \prod_{j=1}^{\infty} f_j(t)$  is a characteristic function and it is positive in a neighbourhood of zero. Let  $t_0$  be the smallest value of  $t > 0$  for which  $\phi_1(t_0) = 0$ . Since  $\phi_1(t) = \prod_{j=1}^{\infty} f_j(t)$ , it follows that  $t_0$  is a zero of at least one of the functions, say,  $f_k(t)$ . Observe that, for all  $|t| < t_0$ , the inequality  $\prod_{j=1}^{\infty} f_j(t) > 0$  holds. Since the sequence  $c_j, j \geq 1$  is bounded, for sufficiently small  $s$ , say,  $|s| < \epsilon$ , all the functions  $f_j(c_j s)$  are strictly positive. Therefore

$$\prod_{j=1}^{\infty} f_j(t) \prod_{j=1}^{\infty} f_j(c_j s) > 0$$

for  $|t| < t_0, |s| < \epsilon$ . However, at the same time, there exists  $t_1 < t_0$  and  $s_1, |s_1| < \epsilon$  for which  $t_1 + c_k s_1 = t_0$  such that

$$\prod_{j=1}^{\infty} f_j(t_1) \prod_{j=1}^{\infty} f_j(c_j s_1) = \prod_{j=1}^{\infty} f_j(t_1 + c_j s_1) \exp[q(t_1, s_1)].$$

Note that the quantity on the left side of the equation is positive whereas the quantity on the right side of the equation is zero leading to a contradiction. Hence the function  $f_j(t) > 0$  for all  $t \in R$  and for all  $j \geq 1$ .

**Proof of Theorem 3.1:** The convergence of the three infinite products in the equation (3.1) is uniform for  $s, t$  in any bounded interval  $I$  and  $f_j(t) \neq 0, f_j(c_j s) \neq 0$  and  $f_j(t + c_j s) \neq 0$  for all  $t, s$  in  $I$  and for all  $j \geq 1$ . Let  $\psi_j(t) = \log f_j(t), j \geq 1$ . Note that  $\psi_j(0) = \log f_j(0) = 0$ . The equation (3.1) can be written in the form

$$(3. 2) \quad \sum_{j=1}^{\infty} \psi_j(t) + \sum_{j=1}^{\infty} \psi_j(c_j s) = \sum_{j=1}^{\infty} \psi_j(t + c_j s) + q(t, s)$$

where  $q(t, s)$  is a polynomial in  $t, s$  or equivalently

$$(3. 3) \quad \sum_{j=1}^{\infty} \psi_j(t + c_j v) = \sum_{j=1}^{\infty} \psi_j(u) + \sum_{j=1}^{\infty} \psi_j(c_j v) - q(u, v) = A(u) + B(v) + r(u, v) \quad (\text{say})$$

where  $\psi_j, j \geq 1$  are continuous,  $\psi_j(0) = 0$  and the function  $r(u, v)$  is polynomial in  $u, v$ .

We note the following properties of uniformly convergent series all of whose terms are of the same sign. Any sub-series of such a series is uniformly convergent. If the terms of the series are multiplied by a bounded sequence, then the resulting series is also uniformly convergent. We now present the arguments of Ramachandran (1967). Since

$$\psi_j(u + c_j v) = \psi_j(u) + \psi_j(c_j v)$$

trivially if  $c_j = 0$ , we may subtract from the three series in (3.3) their respective subseries corresponding to indices  $j$  for which  $c_j = 0$ . The three resulting series are uniformly convergent. Hence, we assume here after that the summations in (3.3) are over those  $j$  for which  $c_j \neq 0$  and the uniform convergence of the series continues to hold for the three series in the equation (3.3) for  $u, v$  in the interval  $I$ . Multiplying both sides of the above equation by  $(x - u)$  and integrate with respect to  $u$  from 0 to  $x$ , we get that

$$\sum_j \int_0^x \psi_j(u + c_j v)(x - u) du = \int_0^x A(u)(x - u) du + \int_0^x B(v)(x - u) du$$

$$\begin{aligned}
& + \int_0^x r(u, v)(x - u)du \\
= & \int_0^x A(u)(x - u)du + B(v)\frac{x^2}{2} + x^2r_1(x, v) \\
= & C(x) + B(v)\frac{x^2}{2} + x^2r(x, v)
\end{aligned}$$

where  $\{\psi_j, j \geq 1\}$  are continuous,  $\psi_j(0) = 0, j \geq 1$ , and  $r_1(x, v)$  is a polynomial in  $x, v$ . In view of the boundedness of the sequence  $\{c_j, j \geq 1\}$  and the uniform convergence of the series  $\psi_j(c_jv)$  of negative terms, the following equation holds:

$$\begin{aligned}
\sum_j \int_0^x \psi_j(u + c_jv)(x - u)du & = \sum_j \int_{c_jv}^{x+c_jv} \psi_j(t)(x - t + c_jv)dt \\
& = \sum_j \int_0^{x+c_jv} \psi_j(t)(x - t + c_jv)dt \\
& \quad - \sum_j \int_0^{c_jv} \psi_j(t)(x - t + c_jv)dt.
\end{aligned}$$

Hence

$$(3.4) \sum_j \int_0^{x+c_jv} \psi_j(t)(x - t + c_jv)dt = C(x) + B_1(v)x^2 + B_2(v)x + B_3(v) + x^2r_1(x, v)$$

for some function  $r_1(x, v)$  which is a polynomial in  $x, v$ . The integral on the left side of the above equation is differentiable twice with respect to  $v$  for any fixed  $x$ . From the uniform convergence of the series discussed above, it follows that the series on the left side can be formally differentiated once and once again with respect to  $v$ . Both the derived series are uniformly convergent and hence the functions on the right side of the equation (3.4) can be differentiated twice with respect to  $v$  for every  $v \in I$ . Hence the functions  $B_j(v), j = 1, 2, 3$  and the function  $r_1(x, v)$  are differentiable with respect to  $v$ . Differentiating both sides of the equation with respect to  $v$ , we obtain that

$$(3.5) \quad \sum_j c_j \int_0^{x+c_jv} \psi_j(t)dt = B'_1(v)x^2 + B'_2(v)x + B'_3(v) + x^2 \frac{dr_1(x, v)}{dv}$$

and

$$(3.6) \quad \sum_j c_j^2 \psi_j(x + c_jv) = B''_1(v)x^2 + B''_2(v)x + B''_3(v) + x^2 \frac{d^2r_1(x, v)}{dv^2}.$$



Here  $g'(x)$  denotes the derivative of  $g(x)$  with respect to  $x$  and  $g''(x)$  denotes the second derivative of  $g(x)$  with respect to  $x$  whenever they exist. Let  $v = 0$  in the above equation. Then we get that

$$(3. 7) \quad \sum_j c_j^2 \psi_j(x) = P(x) + x^2 \frac{d^2 r_1(x, v)}{dv^2} \Big|_{[v=0]} = R(x)$$

where  $R(x)$  is a polynomial in  $x$ . Hence

$$(3. 8) \quad \prod_j [f_j(t)]^{c_j^2} = e^{R(x)}$$

for  $t \in R$ . Applying Theorem 6.4.2 in Linnik (1964)(cf. Theorem 7.3, Ramachandran (1967)), it follows that the polynomial  $R(x)$  is of degree at most two. Applying the results in Cramer (1936), we get that the functions  $f_j$  are characteristic functions of the normal distribution. If the degree of the polynomial  $R(x)$  is one, then all the characteristic functions  $f_j, j \geq$  are degenerate.

**Remarks 3.1:** Rao (1971) proved that, if  $X_i, 1 \leq i \leq n$  are independent random variables, then one can construct  $p$  linear forms  $L_1, \dots, L_p$  of  $X_1, \dots, X_n$  with  $p(p-1)/2 \leq n \leq p(p+1)/2$  such that the joint distribution of  $L_1, \dots, L_p$  determines the distribution of  $X_i, 1 \leq i \leq n$  up to  $Q$ -identical distributions. This was pointed out in Kagan and Székely (2016).

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