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Parametric Estimation for Linear Stochastic Differential Equations Driven by Sub-Fractional Brownian Motion

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Abstract

We investigate the asymptotic properties of the maximum likelihood estimator and Bayes estimator of the drift parameter for stochastic processes satisfying linear stochastic differential equations driven by a sub-fractional Brownian motion. We obtain a Bernstein-von Mises type theorem also for such a class of processes.

Keywords and phrases: Linear stochastic differential equations ; sub-fractional Ornstein-Uhlenbeck process; sub-fractional Brownian motion; Maximum likelihood estimation; Bayes estimation; Consistency; Asymptotic normality; Bernstein - Von Mises theorem.

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1 Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (2010). There has been a recent interest to study similar problems for stochastic processes driven by a sub-fractional Brownian motion. Bojdecki et al. (2004) introduced a centered Gaussian process $\zeta^H = \{\zeta^H(t), t \geq 0\}$ called *sub-fractional Brownian motion* (sub-fBm) with the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}]$$

where $0 < H < 1$. The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor (2009) introduced a Wiener integral with respect to a sub-fBm. Tudor (2007 a,b, 2008, 2009) discussed some properties related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained in Tudor (2009). Diedhiou et al. (2011) investigated parametric estimation for a stochastic differential equation (SDE) driven by a sub-fBm. Mendy (2013) studied parameter estimation for the sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0$$

where $H > \frac{1}{2}$. This is an analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a sub-fBm $\zeta^H = \{\zeta_t^H, t \geq 0\}$ with Hurst parameter H. Mendy (2013) investigated the problem of estimation of the parameters θ based on the observation $\{X_s, 0 \leq s \leq T\}$ and proved that the least squares estimator estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. Kuang and Xie (2013) studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Kuang and Liu (2016) discussed about the L^2-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. (2011) obtained the Ito's formula for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$. Shen and Yan (2014) studied estimation for the drift of sub-fractional Brownian motion and constructed a class of biased estimators of James-Stein type which dominate the maximum likelihood estimator under the quadratic risk. El Machkouri et al. (2016) investigated the asymptotic properties of the least squares estimator for non-ergodic Ornstein-Uhlenbeck process driven by Gaussian processes, in particular, sub-fractional Brownian motion. In a recent paper, we have investigated optimal estimation of a signal perturbed by a sub-fractional Brownian motion in Prakasa Rao (2017b). Some maximal and integral inequalities for a sub-fBm were derived in Prakasa Rao (2017a). We now study more general classes of stochastic processes satisfying linear stochastic differential equations driven by a sub-fractional Brownian motion and investigate the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_T)$-adapted. Further the natural filtration of a process is understood as the P-completion of the filtration generated by this process.

Let $\zeta^H = \{W_t^H, t \geq 0\}$ be a normalized *sub-fractional Brownian motion* (sub-fBm) with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $\zeta_0^H = 0, E(\zeta_t^H) = 0$ and

$$E(\zeta_s^H \zeta_t^H) = t^{2H} + s^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2. 1}$$

Bojdecki et al. (2004) noted that the process

$$\frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0,$$

where $\{W^H(t), -\infty < t < \infty\}$ is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. They proved the following result concerning properties of a sub-fBm.

Theorem 2.1: *Let $\zeta^H = \{\zeta^H(t), t \geq 0\}$ be a sub-fBm defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$.. Then the following properties hold.*

(i) The process ζ^H is self-similar, that is, for every $a > 0$,

$$\{\zeta^H(at), t \geq 0\} \triangleq \{a^H \zeta^H(t), t \geq 0\}$$

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process ζ^H is not Markov and it is not a semi-martingale.

(iii) For all $s, t \geq 0$, the covariance function $C_H(s,t)$ of the process ζ^H is positive for all $s > 0, t > 0$. Furthermore

$$C_H(s,t) > R_H(s,t) \text{ if } H < \frac{1}{2}$$

and

$$C_H(s,t) < R_H(s,t) \text{ if } H > \frac{1}{2}.$$

(iv) Let $\beta_H = 2 - 2^{2H-1}$. For all $s \geq 0, t \geq 0$,

$$\beta_H (t-s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq (t-s)^{2H}, \text{ if } H > \frac{1}{2}$$

and

$$(t-s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq \beta_H (t-s)^{2H}, \text{ if } H < \frac{1}{2}$$

and the constants in the above inequalities are sharp.

(v) The process ζ^H has continuous sample paths almost surely and, for each $0 < \epsilon < H$ and $T > 0$, there exists a random variable $K_{\epsilon,T}$ such that

$$|\zeta^H(t) - \zeta^H(s)| \leq K_{\epsilon,T}|t-s|^{H-\epsilon}, 0 \leq s,t \leq T.$$

Let $f : [0,T] \to R$ be a measurable function and $\alpha > 0$, and σ and η be real. Define the Erdeyli-Kober-type fractional integral

$$(I^{\alpha}_{T,\sigma,\eta}f)(s) = \frac{\sigma s^{\alpha\eta}}{\Gamma(\alpha)}\int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1}f(t)}{(t^{\sigma}-s^{\sigma})^{1-\alpha}}dt, s \in [0,T], \tag{2. 2}$$

and the function

$$\begin{aligned} n_H(t,s) &= \frac{\sqrt{\pi}}{2^{H-\frac{1}{2}}} I^{H-\frac{1}{2}}_{T,2,\frac{3-2H}{4}}(u^{H-\frac{1}{2}})I_{[0,t)}(s) \\ &= \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{\frac{3}{2}-H}\int_0^t (x^2-s^2)^{H-\frac{3}{2}}dx\ I_{(0,t)}(s). \end{aligned} \tag{2. 3}$$

The following theorem is due to Dzhaparidze and Van Zanten (2004) (cf. Tudor (2009)).

Theorem 2.2: *The following representation holds, in distribution, for a sub-fBm ζ^H:*

$$\zeta^H_t \triangleq c_H \int_0^t n_H(t,s)dW_s, 0 \leq t \leq T \tag{2. 4}$$

where

$$c_H^2 = \frac{\Gamma(2H+1)\ \sin(\pi H)}{\pi} \tag{2. 5}$$

and $\{W_t, t \geq 0\}$ is the standard Brownian motion.

Tudor (2009) has defined integration of a non-random function $f(t)$ with respect to a sub-fBm ζ^H on an interval $[0,T]$ and obtained a representation of this integral as a Wiener integral for a suitable transformed function $\phi_f(t)$ depending on H and T. For details, see Theorem 3.2 in Tudor (2009).

Tudor (2007b) obtained the prediction formula for a sub-fBm. For any $0 < H < 1$, and $0 < a < t$,

$$E[\zeta^H_t|\zeta^H_s, 0 \leq s \leq a] = \zeta^H_a + \int_0^a \psi_{a,t}(u)d\zeta^H_u \tag{2. 6}$$

where

$$\psi_{a,t}(u) = \frac{2\ \sin(\pi(H-\frac{1}{2}))}{\pi}u(a^2-u^2)^{\frac{1}{2}-H}\int_a^t \frac{(z^2-a^2)^{H-\frac{1}{2}}}{z^2-u^2}z^{H-\frac{1}{2}}dz. \tag{2. 7}$$

Let

$$M_t^H = d_H \int_0^t s^{\frac{1}{2}-H} dW_s = \int_0^t k_H(t,s) d\zeta_s^H \tag{2. 8}$$

where

$$d_H = \frac{2^{H-\frac{1}{2}}}{c_H \Gamma(\frac{3}{2}-H)\sqrt{\pi}}, \tag{2. 9}$$

$$k_H(t,s) = d_H s^{\frac{1}{2}-H} \psi_H(t,s), \tag{2. 10}$$

and

$$\begin{aligned}\psi_H(t,s) &= \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)}[t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - \\ &\quad (H-\frac{3}{2})\int_s^t (x^2-s^2)^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx] I_{(0,t)}(s).\end{aligned}$$

It can be shown that the process $M^H = \{M_t^H, t \geq 0\}$ is a Gaussian martingale (cf. Tudor (2009), Diedhiou et al. (2011)) and is called the *sub-fractional fundamental martingale*. The filtration generated by this martingale is the same as the filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the sub-fBm ζ^H and the quadratic variation $< M^H >_s$ of the martingale M^H over the interval $[0,s]$ is equal to $w_s^H = \frac{d_H^2}{2-2H} s^{2-2H} = \lambda_H s^{2-2H}$ (say). For any measurable function $f : [0,T] \to R$ with $\int_0^T f^2(s) s^{1-2H} ds < \infty$, define the probability measure Q_f by

$$\begin{aligned}\frac{dQ_f}{dP}|_{\mathcal{F}_t} &= \exp(\int_0^t f(s) dM_s^H - \frac{1}{2}\int_0^t f^2(s) d < M^H > (s)) \\ &= \exp(\int_0^t f(s) dM_s^H - \frac{d_H^2}{2}\int_0^t f^2(s) s^{1-2H} ds).\end{aligned}$$

where P is the underlying probability measure. Let

$$(\psi_H f)(s) = \frac{1}{\Gamma(\frac{3}{2}-H)} I_{0,2,\frac{1}{2}-H}^{H-\frac{1}{2}} f(s) \tag{2. 11}$$

where, for $\alpha > 0$,

$$(I_{0,\sigma,\eta}^{\alpha} f)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^s \frac{t^{\sigma(1+\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0,T]. \tag{2. 12}$$

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor (2009)).

Theorem 2.3: *The process*

$$\zeta_t^H - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T$$

is a sub-fbm with respect to the probability measure Q_f. In particular, choosing the function $f \equiv a \in R$, it follows that the process $\{\zeta_t^H - at, 0 \leq t \leq T\}$ is a sub-fBm under the probability measure Q_f with $f \equiv a \in R$.

Let $Y = \{Y_t, t \geq 0\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ and suppose the process Y satisfies the stochastic differential equation

$$dY_t = C(t)dt + d\zeta_t^H, t \geq 0 \tag{2. 13}$$

where the process $\{C(t), t \geq 0\}$, adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, such that the process

$$R_H(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s)C(s)ds, t \geq 0 \tag{2. 14}$$

is well-defined and the derivative is understood in the sense of absolute continuity with respect to the measure generated by the function w_H. Differentiation with respect to w_t^H is understood in the sense:

$$dw_t^H = \lambda_H(2-2H)t^{1-2H}dt$$

and

$$\frac{df(t)}{dw_t^H} = \frac{df(t)}{dt} / \frac{dw_t^H}{dt}.$$

Suppose the process $\{R_H(t), t \geq 0\}$, defined over the interval $[0,T]$ belongs to the space $L^2([0,T], dw_t^H)$. Define

$$\Lambda_H(t) = \exp\{\int_0^t R_H(s)dM_s^H - \frac{1}{2}\int_0^t [R_H(s)]^2 dw_s^H\}. \tag{2. 15}$$

with $E[\Lambda_H(T)] = 1$ and the distribution of the process Y with respect to the measure $P^Y = \Lambda_t^H P$ coincides with the distribution of the $\int_0^t d\zeta_t^H$ with respect to the measure P.

We call the process Λ^H as the *likelihood process* or the Radon-Nikodym derivative $\frac{dP^Y}{dP}$ of the measure P^Y with respect to the measure P.

Tudor (2009) derived the following Girsanov type formula.

Theorem 2.4: *Suppose the assumptions of Theorem 2.2 hold. Define*

$$\Lambda_H(T) = \exp\{\int_0^T R_H(t)dM_t^H - \frac{1}{2}\int_0^t R_H^2(t)dw_t^H\}. \tag{2. 16}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

(2. 17)
$$V_t = \int_0^t d\zeta_s^H, 0 \leq t \leq T.$$

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3 Main Results

Let us consider the stochastic differential equation

(3. 1)
$$dX(t) = [a(t, X(t)) + \theta\, b(t, X(t))]dt + d\zeta_t^H, t \geq 0$$

where $\theta \in \Theta \subset R, \zeta^H = \{\zeta_t^H, t \geq 0\}$ is a sub-fractional Brownian motion with Hurst parameter H. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

(3. 2)
$$X(t) = X(0) + \int_0^t [a(s, X(s)) + \theta\, b(s, X(s))]ds + \int_0^t d\zeta_s^H, t \geq 0.$$

Let

(3. 3)
$$C(\theta, t) = a(t, X(t)) + \theta\, b(t, X(t)), t \geq 0$$

and assume that the sample paths of the process $\{C(\theta, t), t \geq 0\}$ are smooth enough so that the the process

(3. 4)
$$R_{H,\theta}(t) = \frac{d}{dw_t^H} \int_0^t k_H(t, s)C(\theta, s)ds, t \geq 0$$

is well-defined where w_t^H and $k_H(t, s)$ are as defined above. Suppose the sample paths of the process $\{R_{H,\theta}, 0 \leq t \leq T\}$ belong almost surely to $L^2([0, T], dw_t^H)$. Define

(3. 5)
$$Z_t = \int_0^t k_H(t, s)dX_s, t \geq 0.$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

(3. 6)
$$Z_t = \int_0^t R_{H,\theta}(s)dw_s^H + M_t^H, t \geq 0$$

where M^H is the fundamental martingale and the process X admits the representation

(3. 7)
$$X_t = \int_0^t K_H(t, s)dZ_s$$

where the function

$$K_H(t,s) = \frac{C(H)}{d_H} s^{H-\frac{1}{2}} n_H(t,s).$$

Let P_θ^T be the measure induced by the process $\{X_t, 0 \le t \le T\}$ when θ is the true parameter. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[\int_0^T R_{H,\theta}(s) dZ_s - \frac{1}{2}\int_0^T R_{H,\theta}^2(s) dw_s^H]. \tag{3. 8}$$

Maximum likelihood estimation

We now consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \le t \le T\}$ and study its asymptotic properties as $T \to \infty$.

Strong consistency:

Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta). \tag{3. 9}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao (1987)).

Note that

$$\begin{aligned} R_{H,\theta}(t) &= \frac{d}{dw_t^H}\int_0^t k_H(t,s) C(\theta,s) ds \\ &= \frac{d}{dw_t^H}\int_0^t k_H(t,s) a(s,X(s)) ds + \theta \frac{d}{dw_t^H}\int_0^t k_H(t,s) b(s,X(s)) ds \\ &= J_1(t) + \theta J_2(t).(say) \end{aligned} \tag{3. 10}$$

Then

$$\log L_T(\theta) = \int_0^T (J_1(t) + \theta J_2(t)) dZ_t - \frac{1}{2}\int_0^T (J_1(t) + \theta J_2(t))^2 dw_t^H \tag{3. 11}$$

and the likelihood equation is given by

$$\int_0^T J_2(t) dZ_t - \int_0^T (J_1(t) + \theta J_2(t)) J_2(t) dw_t^H = 0. \tag{3. 12}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

(3. 13) $$\hat{\theta}_T = \frac{\int_0^T J_2(t)dZ_t + \int_0^T J_1(t)J_2(t)dw_t^H}{\int_0^T J_2^2(t)dw_t^H}.$$

Let θ_0 be the true parameter. Using the fact that

(3. 14) $$dZ_t = (J_1(t) + \theta_0 J_2(t))dw_t^H + dM_t^H,$$

it can be shown that

(3. 15) $$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp[(\theta - \theta_0)\int_0^T J_2(t)dM_t^H - \frac{1}{2}(\theta - \theta_0)^2 \int_0^T J_2^2(t)dw_t^H].$$

Following this representation of the Radon-Nikodym Derivative, we obtain that

(3. 16) $$\hat{\theta}_T - \theta_0 = \frac{\int_0^T J_2(t)dM_t^H}{\int_0^T J_2^2(t)dw_t^H}.$$

We now discuss the problem of estimation of the parameter θ on the basis of the observation of the process X or equivalently the process Z on the interval $[0, T]$.

Theorem 3.1: *The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,*

(3. 17) $$\hat{\theta}_T \to \theta_0 \;\; a.s \;\; [P_{\theta_0}] \;\; as \;\; T \to \infty$$

provided

(3. 18) $$\int_0^T J_2^2(t)dw_t^H \to \infty \;\; \text{a.s} \;\; [P_{\theta_0}] \;\; \text{as} \;\; T \to \infty.$$

Proof: This theorem follows by observing that the process

(3. 19) $$\gamma_T \equiv \int_0^T J_2(t)dM_t^H, t \geq 0$$

is a local martingale with the quadratic variation process

(3. 20) $$< \gamma_T > = \int_0^T J_2^2(t)dw_t^H$$

and applying the Strong law of large numbers (cf. Liptser (1980); Liptser and Shiryayev (1989); Prakasa Rao (1999), p. 61) under the condition (3.18) stated above.

Remark: For the case sub-fractional Ornstein-Uhlenbeck process investigated in Mendy (2013), it can be checked that the condition stated in equation (3.18) holds and hence the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Limiting distribution:

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2: *Assume that the functions $b(t,s)$ and $\sigma(t)$ are such that the process $\{\gamma_t, t \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_t, t \geq 0$ such that*

$$I_T^2 < \gamma_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H \to \eta^2 \text{ in probability as } T \to \infty \tag{3. 21}$$

where $I_T \to 0$ as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_T \gamma_T, I_T^2 < \gamma_T >) \to (\eta Z, \eta^2) \text{ in law as } T \to \infty \tag{3. 22}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Proof: This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao (1999), p. 65).

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T \gamma_T}{I_T^2 < \gamma_T >} \tag{3. 23}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3: *Suppose the conditions stated in the Theorem 3.2 hold. Then*

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in law as } t \to \infty \tag{3. 24}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

Bayes estimation

Suppose that the parameter space Θ is open and Λ is a prior probability measure on the parameter space Θ. Further suppose that Λ has the density $\lambda(.)$ with respect to the Lebesgue measure and the density function is continuous and positive in an open neighbourhood of θ_0, the true parameter. Let

(3. 25) $$\alpha_T \equiv I_T V_T = I_T \int_0^T J_2(t) dM_t^H$$

and

(3. 26) $$\beta_T \equiv I_T^2 < V_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H.$$

We have seen earlier that the maximum likelihood estimator satisfies the relation

(3. 27) $$\alpha_T = (\hat{\theta}_T - \theta_0) I_T^{-1} \beta_T.$$

The posterior density of θ given the observation $X^T \equiv \{X_s, 0 \le s \le T\}$ is given by

(3. 28) $$p(\theta|X^T) = \frac{\frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)}{\int_\Theta \frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)d\theta}.$$

Let us write $t = I_T^{-1}(\theta - \hat{\theta}_T)$ and define

(3. 29) $$p^*(t|X^T) = I_T \;\; p(\hat{\theta}_T + tI_T|X^T).$$

Then the function $p^*(t|X^T)$ is the posterior density of the transformed variable $t = I_T^{-1}(\theta - \hat{\theta}_T)$. Let

(3. 30) $$\begin{aligned} \nu_T(t) &\equiv \frac{dP_{\hat{\theta}_T + tI_T}/dP_{\theta_0}}{dP_{\hat{\theta}_T}/dP_{\theta_0}} \\ &= \frac{dP_{\hat{\theta}_T + tI_T}}{dP_{\hat{\theta}_T}} a.s. \end{aligned}$$

and

(3. 31) $$C_T = \int_{-\infty}^{\infty} \nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt.$$

It can be checked that

(3. 32) $$p^*(t|X^T) = C_T^{-1}\nu_T(t)\lambda(\hat{\theta}_T + tI_T).$$

Furthermore, the equations (3.15) and (3.27)-(3.32) imply that

(3. 33) $$\begin{aligned} \log \nu_T(t) &= I_T^{-1}\alpha_T[(\hat{\theta}_T + tI_T - \theta_0) - (\hat{\theta}_T - \theta_0)] \\ &\quad -\frac{1}{2} I_T^{-2}\beta_T[(\hat{\theta}_T + tI_T - \theta_0)^2 - (\hat{\theta}_T - \theta_0)^2] \\ &= t\alpha_T - \frac{1}{2}t^2\beta_T - t\beta_T I_T^{-1}(\hat{\theta}_T - \theta_0) \\ &= -\frac{1}{2}\beta_T t^2 \end{aligned}$$

in view of equation (3.27).

Suppose that the convergence in the condition in the equation (3.21) holds almost surely under the measure P_{θ_0} and the limit is a constant $\eta^2 > 0$ with probability one. For convenience, we write $\beta = \eta^2$. Then

(3. 34) $$\beta_T \to \beta \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty.$$

Then it is obvious that

(3. 35) $$\lim_{T\to\infty} \nu_T(t) = \exp[-\frac{1}{2}\beta t^2] \text{ a.s. } [P_{\theta_0}]$$

and for any $0 < \varepsilon < \beta$,

(3. 36) $$\log \nu_T(t) \leq -\frac{1}{2}t^2(\beta - \varepsilon)$$

for every t for T sufficiently large. Further more , for every $\delta > 0$, there exists $\varepsilon' > 0$ such that

(3. 37) $$\sup_{|t|>\delta I_T^{-1}} \nu_T(t) \leq \exp[-\frac{1}{4}\varepsilon' I_T^{-2}]$$

for T sufficiently large.

Suppose that $H(t)$ is a nonnegative measurable function such that, for some $0 < \varepsilon < \beta$,

(3. 38) $$\int_{-\infty}^{\infty} H(t)\exp[-\frac{1}{2}t^2(\beta - \varepsilon)]dt < \infty.$$

Suppose the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

(3. 39) $$\hat{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty.$$

For any $\delta > 0$, consider

$$\begin{aligned}
&\int_{|t|\leq\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\
&\leq \int_{|t|\leq\delta I_T^{-1}} H(t)\lambda(\theta_0)|\nu_T(t) - \exp(-\frac{1}{2}\beta t^2)|dt \\
&+ \int_{|t|\leq\delta I_T^{-1}} H(t)\nu_T(t)|\lambda(\theta_0) - \lambda(\hat{\theta}_T + tI_T)|dt
\end{aligned}$$

(3. 40) $$= A_T + B_T(say).$$

It is clear that, for any $\delta > 0$,

(3. 41) $$A_T \to 0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty$$

by the dominated convergence theorem in view of the inequality in (3.36), the equation (3.35) and the condition in the equation (3.38). On the other hand, for T sufficiently large,

$$0 \leq B_T \leq \sup_{|\theta-\theta_0|\leq\delta} |\lambda(\theta) - \lambda(\theta_0)| \int_{|t|\leq\delta I_T^{-1}} H(t)\exp[-\frac{1}{2}t^2(\beta-\varepsilon)]dt \tag{3. 42}$$

since $\hat{\theta}_T$ is strongly consistent and $I_T^{-1} \to \infty$ as $T \to \infty$. The last term on the right side of the above inequality can be made smaller than any given $\rho > 0$ by choosing δ sufficiently small in view of the continuity of $\lambda(.)$ at θ_0. Combining these remarks with the equations (3.41) and (3.42), we obtain the following lemma.

Lemma 3.4: *Suppose the conditions (3.34), (3.38) and (3.39) hold. Then there exists $\delta > 0$ such that*

$$\lim_{T\to\infty} \int_{|t|\leq\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt = 0. \tag{3. 43}$$

For any $\delta > 0$, consider

$$\begin{aligned} &\int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\ &\leq \int_{|t|>\delta I_T^{-1}} H(t)\nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt \\ &\quad + \int_{|t|>\delta I_T^{-1}} H(t)\lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)dt \\ &\leq \exp[-\frac{1}{4}\varepsilon' I_T^{-2}] \int_{|t|>\delta I_T^{-1}} H(t)\lambda(\hat{\theta}_T + tI_T)dt \\ &\quad + \lambda(\theta_0)\int_{|t|>\delta I_T^{-1}} H(t)\exp(-\frac{1}{2}\beta t^2)dt \\ &= U_T + V_T (say). \end{aligned} \tag{3. 44}$$

Suppose the following condition holds for every $\varepsilon > 0$ and $\delta > 0$:

$$\exp[-\varepsilon I_T^{-2}] \int_{|u|>\delta} H(uI_T^{-1})\lambda(\hat{\theta}_T + u)du \to 0 \text{ a.s.}[P_{\theta_0}] \text{ as } T \to \infty. \tag{3. 45}$$

It is clear that, for every $\delta > 0$,

$$V_T \to 0 \text{ as } T \to \infty \tag{3. 46}$$

in view of the condition stated in (3.38) and the fact that $I_T \to \infty$ a.s. $[P_{\theta_0}]$ as $T \to \infty$. The condition stated in (3.45) implies that

$$U_T \to 0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty \tag{3. 47}$$

for every $\delta > 0$. Hence we have the following lemma.

Lemma 3.5: *Suppose that the conditions (3.34), (3.38) and (3.39) hold. Then for every* $\delta > 0$,

(3. 48) $$\lim_{T\to\infty} \int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2))|dt = 0.$$

Lemmas 3.4 and 3.5 together prove that

(3. 49) $$\lim_{T\to\infty} \int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt = 0.$$

Let $H(t) \equiv 1$. It follows that

$$C_T \equiv \int_{-\infty}^{\infty} \nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt.$$

Relation (3.49) implies that

(3. 50) $$C_T \to \lambda(\theta_0)\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\beta t^2)dt = \lambda_{\theta_0}(\frac{\beta}{2\pi})^{-1/2} \text{ a.s}[P_{\theta_0}]$$

as $T \to \infty$. Further more

(3. 51) $$\begin{aligned}&\int_{-\infty}^{\infty} H(t)|p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt \\ &\quad\leq \int_{-\infty}^{\infty} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\ &\quad\quad+ \int_{-\infty}^{\infty} H(t)|C_T^{-1}\lambda(\theta_0) - (\frac{\beta}{2\pi})^{1/2}|\exp(-\frac{1}{2}\beta t^2)dt.\end{aligned}$$

The last two terms tend to zero almost surely $[P_{\theta_0}]$ by the equations (3.49) and (3.50). Hence we have the following theorem which is an analogue of the Bernstein - von Mises theorem proved in Prakasa Rao (1981) for a class of processes satisfying a linear stochastic differential equation driven by the standard Wiener process.

Theorem 3.6: *Let the assumptions (3.34),(3.38),(3.39) and (3.45) hold where* $\lambda(.)$ *is a prior density which is continuous and positive in an open neighbourhood of* θ_0, *the true parameter. Then*

(3. 52) $$\lim_{T\to\infty} \int_{-\infty}^{\infty} H(t)|p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt = 0 \text{ a.s } [P_{\theta_0}].$$

As a consequence of the above theorem, we obtain the following result by choosing $H(t) = |t|^m$, for integer $m \geq 0$.

Theorem 3.7: *Assume that the following conditions hold:*

$$(C1)\ \hat{\theta}_T \to \theta_0 \ \ a.s\ [P_{\theta_0}]\ \ as\ T \to \infty, \tag{3. 53}$$

$$(C2)\ \beta_T \to \beta > 0\ \ a.s\ [P_{\theta_0}]\ \ as\ T \to \infty. \tag{3. 54}$$

Further suppose that
(C3)$\lambda(.)$ is a prior probability density on Θ which is continuous and positive in an open neighbourhood of θ_0, the true parameter and

$$(C4)\ \int_{-\infty}^{\infty} |\theta|^m \lambda(\theta) d\theta < \infty \tag{3. 55}$$

for some integer $m \geq 0$. Then

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |t|^m |p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2} \exp(-\frac{1}{2}\beta t^2)| dt = 0 \ \text{ a.s } [P_{\theta_0}]. \tag{3. 56}$$

In particular, choosing $m = 0$, we obtain that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2} \exp(-\frac{1}{2}\beta t^2)| dt = 0 \ \text{ a.s } [P_{\theta_0}] \tag{3. 57}$$

whenever the conditions (C1), (C2) and (C3) hold. This is the analogue of the Bernstein-von Mises theorem for a class of diffusion processes proved in Prakasa Rao (1981) and it shows the asymptotic convergence in L_1-mean of the posterior density to the normal distribution.

As a Corollary to Theorem 3.7, we also obtain that the conditional expectation, under P_{θ_0}, of $[I_T^{-1}(\hat{\theta}_T - \theta)]^m$ converges to the corresponding m-th absolute moment of the normal distribution with mean zero and variance β^{-1}.

We define a *regular Bayes estimator* of θ, corresponding to a prior probability density $\lambda(\theta)$ and the loss function $L(\theta, \phi)$, based on the observation X^T, as an estimator which minimizes the posterior risk

$$B_T(\phi) \equiv \int_{-\infty}^{\infty} L(\theta, \phi) p(\theta|X^T) d\theta. \tag{3. 58}$$

over all the estimators ϕ of θ. Here $L(\theta, \phi)$ is a loss function defined on $\Theta \times \Theta$.

Suppose there exists a measurable regular Bayes estimator $\tilde{\theta}_T$ for the parameter θ (cf. Theorem 3.1.3, Prakasa Rao (1987).) Suppose that the loss function $L(\theta, \phi)$ satisfies the following conditions:

$$L(\theta, \phi) = \ell(|\theta - \phi|) \geq 0 \tag{3. 59}$$

and the function $\ell(t)$ is nondecreasing for $t \geq 0$. An example of such a loss function is $L(\theta, \phi) = |\theta - \phi|$. Suppose there exist nonnegative functions $S(t)$, $K(t)$ and $G(t)$ such that

$$(D1) \;\; S(t)\ell(tI_T) \leq G(t) \;\; \text{for all} \;\; T \geq 0, \tag{3. 60}$$

$$(D2) \;\; S(t)\ell(tI_T) \to K(t) \;\; \text{as} \;\; T \to \infty \tag{3. 61}$$

uniformly on bounded intervals of t. Further suppose that the function

$$(D3) \;\; \int_{-\infty}^{\infty} K(t+h) \exp[-\frac{1}{2}\beta t^2] dt \tag{3. 62}$$

has a strict minimum at $h = 0$, and
(D4)the function $G(t)$ satisfies the conditions similar to (3.38) and (3.45).

We have the following result giving the asymptotic properties of the Bayes risk of the estimator $\tilde{\theta}_T$.

Theorem 3.8: *Suppose the conditions (C1) to (C3) in the Theorem 3.6 and the conditions (D1) to (D4) stated above hold. Then*

$$I_T^{-1}(\tilde{\theta}_T - \hat{\theta}_T) \to 0 \;\; a.s \;\; [P_{\theta_0}] \;\; as \;\; T \to \infty \tag{3. 63}$$

and

$$\begin{aligned} \lim_{T\to\infty} S(T) B_T(\tilde{\theta}_T) &= \lim_{T\to\infty} S(T) B_T(\hat{\theta}_T) \\ &= (\frac{\beta}{2\pi})^{1/2} \int_{-\infty}^{\infty} K(t) \exp[-\frac{1}{2}\beta t^2] dt \;\; \text{a.s} \;\; [P_{\theta_0}] \end{aligned} \tag{3. 64}$$

We omit the proof of this theorem as it is similar to the proof of Theorem 4.1 in Borwanker et al. (1971).

We have observed earlier that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \;\; \text{in law} \;\; \text{as} \;\; T \to \infty. \tag{3. 65}$$

As a consequence of the Theorem 3.8, we obtain that

$$\tilde{\theta}_T \to \theta_0 \;\; \text{a.s} \;\; [P_{\theta_0}] \;\; \text{as} \;\; T \to \infty \tag{3. 66}$$

and

$$I_T^{-1}(\tilde{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \;\; \text{in law} \;\; \text{as} \;\; T \to \infty. \tag{3. 67}$$

In other words, the Bayes estimator is asymptotically normal and has asymptotically the same distribution as the maximum likelihood estimator. The asymptotic Bayes risk of the estimator is given by the Theorem 3.7.

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